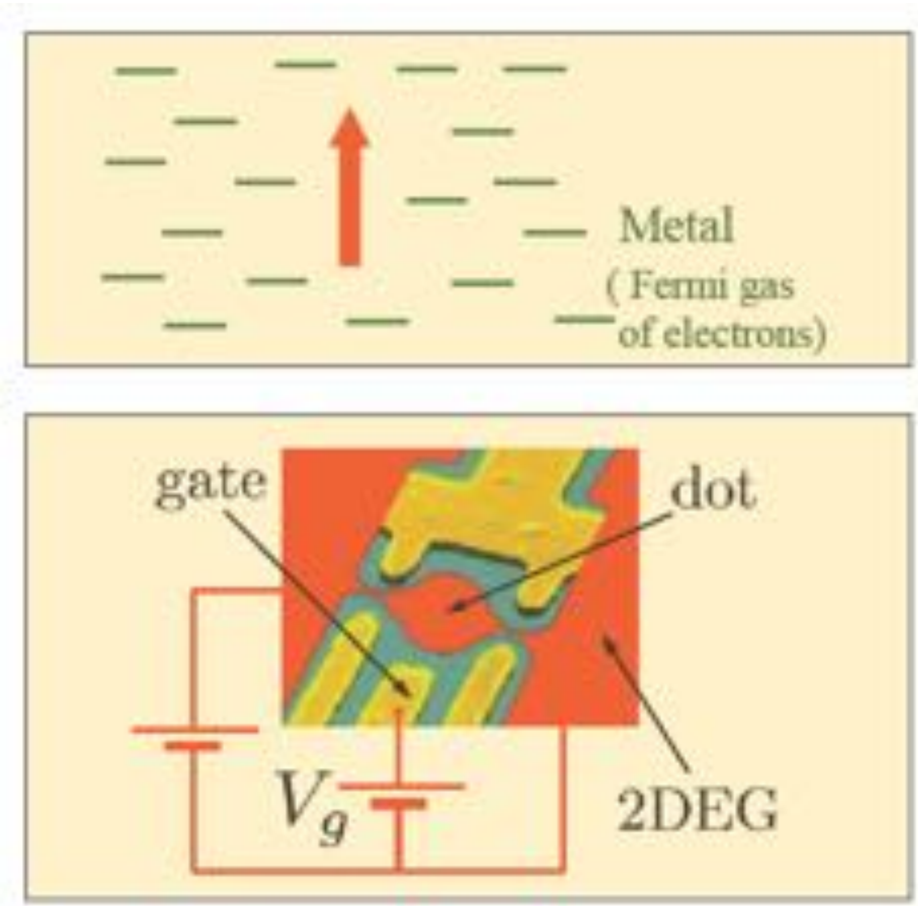
The Kondo problem: Bethe Ansatz Solution and underlying scaling physics



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Ref: N. Andrei in Series on Modern Condensed Matter Physics - Vol. 6, World Scientic, Lecture Notes of ICTP Summer Course. Editors: S. Lundquist, G. Morandi and Yu Lu. arXiv: 9408101

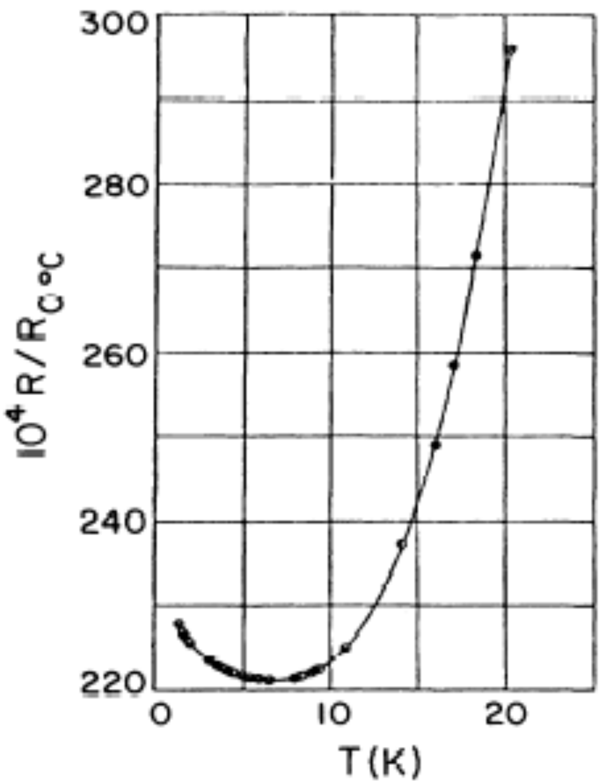
Autumn School on Correlated Electrons, Julich - Sept 2015

Resistivity minimum

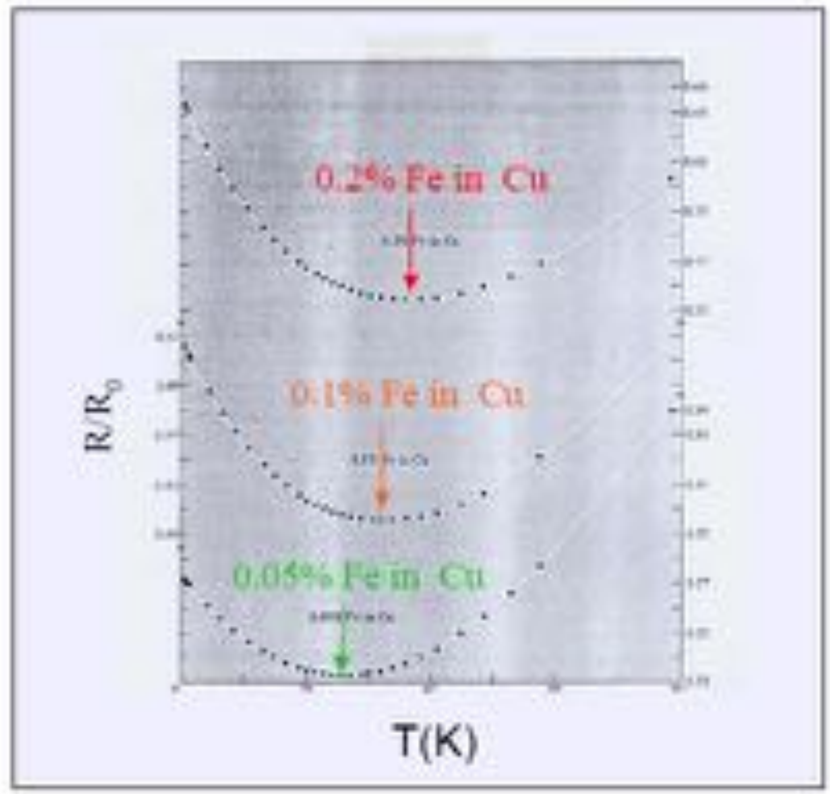
Measurements of electric resistivity of a metal with dilute concentration of magnetic impurities:

Enhanced scattering at low T

$T_{\min} \propto c_{\text{imp}}^{1/5}$



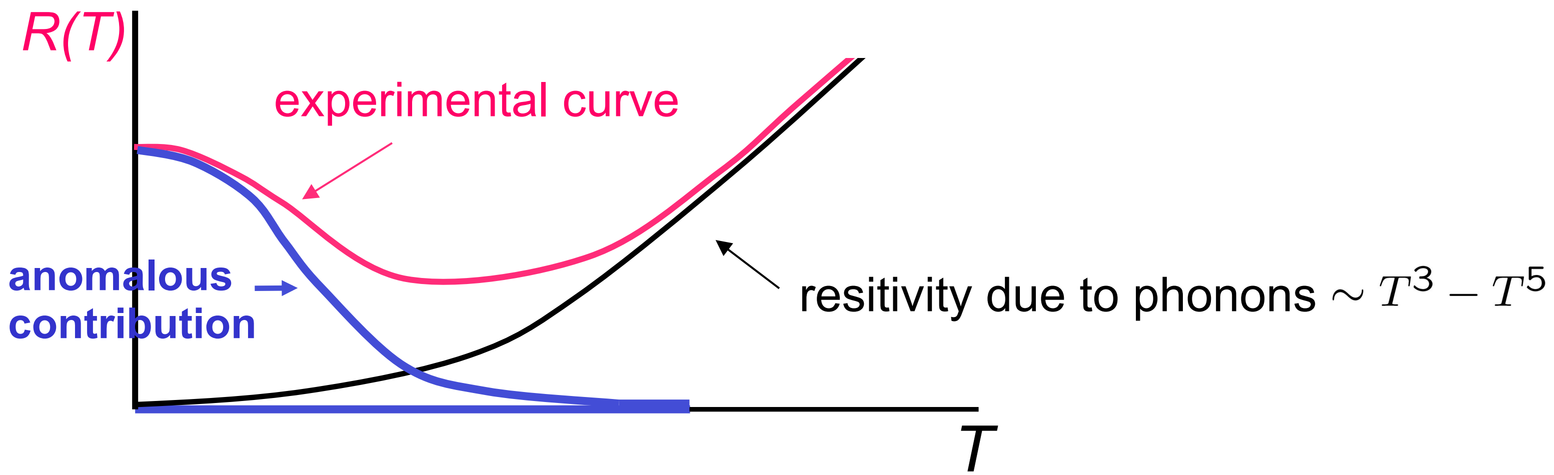
De Haas & ven den Berg, 1936



Franck *et al*., 1961

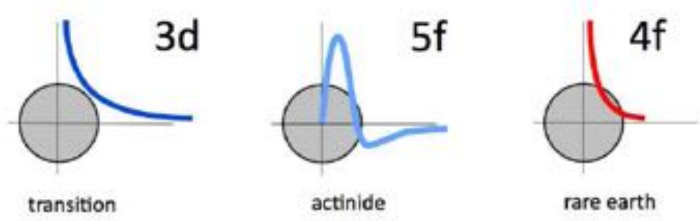
The Kondo Effect – resistivity minimum

- upturn of *R(T)* at low-T, as opposed to the pure metal behavior:



- Why is scattering stronger at low temperatures, weaker at high temperatures ?
- Source of extra scattering? ... anomalous scattering due to ***magnetic impurities***?

- **local magnetic moment** : the spin of unpaired electrons in atomic ***d*** or ***f*** shell.



e.g. Fe U Ce

The (s-d) Kondo Hamiltonian

- **Electrons in the presence of dilute *magnetic* impurities**

  - **Conduction band of electrons** (metal)

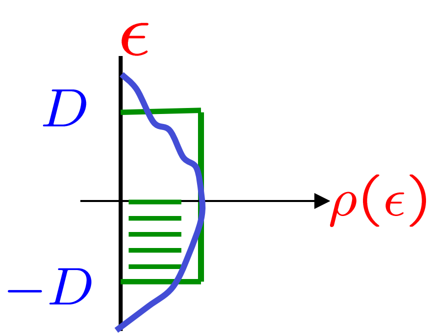
$$H_o = \sum_k \epsilon_k c^*_{ka} c_{ka}$$

  - **Exchange interaction -**
    electron spin density $\sigma_{el}(\vec{x}=0)$ with impurity spin S **:**

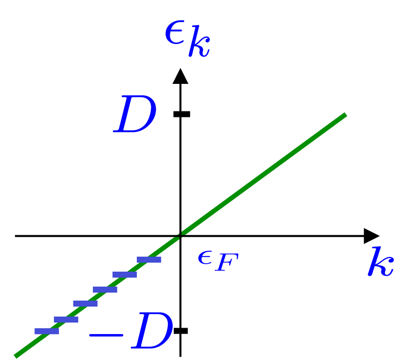
$$\begin{aligned} H_1 &= J\,\sigma_{el}(\vec{x}=0)\cdot\vec{S} = J\sum_{ka}\sum_{k'a'} c^*_{ka}\vec{\sigma}_{aa'}c_{k'a'}\cdot\vec{S} \\ &= J\sum_k\sum_{k'}[c^*_{k\uparrow}c_{k'\downarrow}S^- + c^*_{k\downarrow}c_{k'\uparrow}S^+ + (c^*_{k\uparrow}c_{k'\uparrow} - c^*_{k\downarrow}c_{k'\downarrow})S^z] \end{aligned}$$

  - **Antiferromagnetic** coupling $J > 0$

- **Flat band approximation:** **(low-E, universal results)**



- linearize spectrum $\epsilon_k = v_F k$
- band-width $D, \; -D \le k \le D$



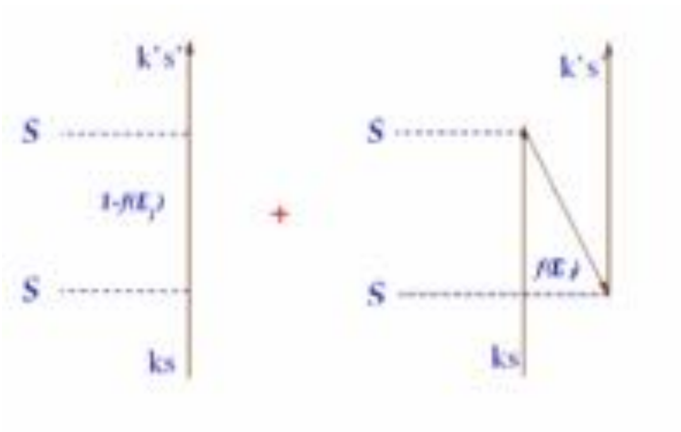
The resistivity (Kondo ’64)

• Scattering amplitude : electron with momentum *k* and spin$\downarrow$ into state with momentum *k*’, impurity remains with spin$\Uparrow$:

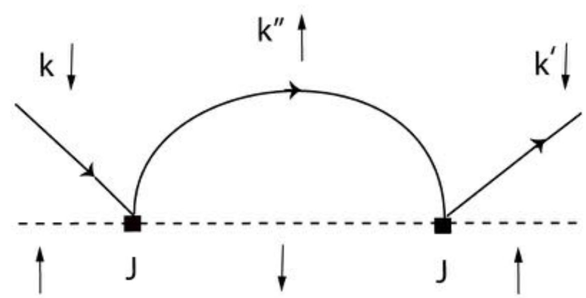
- first order $\quad J(k\downarrow,\Uparrow\rightarrow k'\downarrow,\Uparrow) = J$

- second order $\quad \sum_{k''} J(k\downarrow,\Uparrow\rightarrow k''\uparrow,\Downarrow)J(k''\uparrow,\Downarrow\rightarrow k'\downarrow,\Uparrow)\frac{1-f_{k''}}{\epsilon_k-\epsilon_{k''}}+..$

Flat band approx. $\quad = J^2\rho\int_{\epsilon_F}^{D}\frac{1}{\epsilon_k-\epsilon_{k''}}d\epsilon_{k''} = J^2\rho\,\mathsf{log}\left(\left|\frac{\epsilon_k-\epsilon_F}{\epsilon_k-D}\right|\right)$



or



• Scattering probability $\quad W_k = J^2 + 2J^3\rho\,\mathsf{log}\left(\left|\frac{\epsilon_k-\epsilon_F}{\epsilon_k-D}\right|\right)$

The resitivity (Kondo '64)

Finite temperature resistivity –

consider electron with energy within a window $k_B T$ about the Fermi energy:

$$R(T) = R_0 \left[1 + 2J\rho \log\left(\left| \frac{D-\epsilon_F}{k_B T} \right|\right)\right]$$

- For $J > 0$ resistivity increases as T decreases
- Combine with phonon contribution to account for resistivity minimum
- Correction small at high temperatures but diverges as $T \searrow 0$
- The n-order ~ $[J\rho \log(D/T)]^n$ diverges, resummation does not help.
- Perturbation theory breaks down at T such that $J\rho \log(D/T) \sim 1$

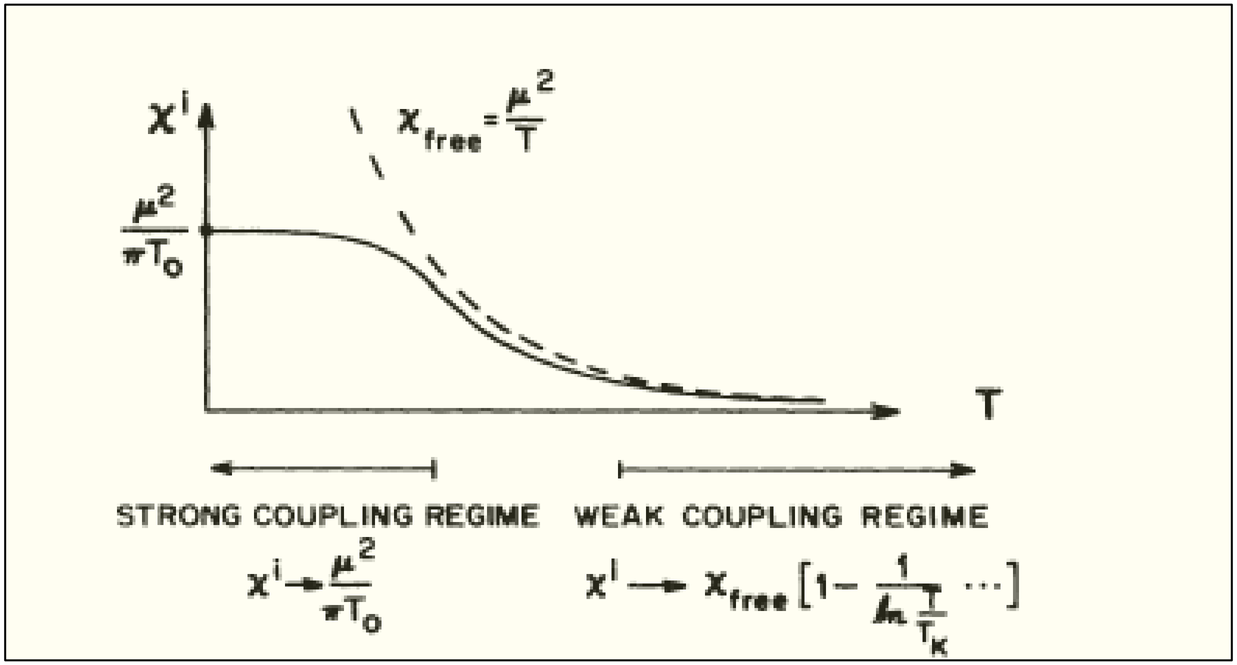
\- Kondo temperature: $T_K = De^{-1/J\rho}$

- What to do for $T \leq T_K$? The **Kondo Problem**

Magnetic susceptibility

- We saw, considering resistivity, that the effect was weak at high temperatures, strong at low temperatures

- Similar behavior in impurity magnetic susceptibility: $\chi^i = \chi_{total} - \chi_{metal}$



$$M_{\rm free} = \frac{\mu\, e^{\mu h/T} - \mu\, e^{-\mu h/T}}{e^{\mu h/T} + e^{-\mu h/T}}$$

$$\chi_{\rm free} = \frac{\partial M}{\partial h}\Big|_{h=0} = \frac{\mu^2}{T}$$

- **At high temperatures - free spin susceptibility**

$$\chi^i \to \chi_{\rm free} = \frac{\mu^2}{T}$$

- **At low temperatures - spin is screened**

$$\chi^i \to \text{finite}$$

The Kondo Effect

- Resumming leading logs $(J\rho)^n(\log(D/T))^{(n-1)}$ **of perturbation theory**

$$\begin{aligned}\chi^i(T) &= \frac{\mu^2}{T}[1 - J + J^2 \log\frac{T}{D} - J^3 \log^2\frac{T}{D} + \cdots] \\ &= \frac{\mu^2}{T}[1 - \frac{J}{1 + J\log\frac{T}{D}} + \cdots] \\ &= \frac{\mu^2}{T}[1 - \frac{1}{\log\frac{T}{T_K}} + \cdots]\end{aligned}$$

only for *J>0* does it diverges as *T* is lowered

where: $T_K = De^{-1/J}$

- Resumming sub-leading logs $(J\rho)^n(\log(D/T))^{(n-2)}$

$$\chi^i(T) = \frac{\mu^2}{T}\left[1 - \frac{1}{\log\frac{T}{T_K}} - \frac{1}{2}\frac{\log(\log(T/T_K))}{\log^2(T/T_K)} + \cdots\right]$$

- Perturbation theory breaks down at low temperatures (i.e. in the IR)
- Perturbation theory valid at high temperatures
- A new low energy scale appears: $T_K = De^{-1/J}$
- Strong coupling IR, weak coupling UV
- **How to handle such a theory? –** ***The Kondo Problem***

The Kondo Problem

Many approaches to the Kondo Problem:

- Resummation of the perturbation series (fails)
- Variational techniques (fail)

- Scaling theory – Poor Man's Scaling (P. W. Anderson)
- Renormalization group (K. Wilson)

- Fermi Liquid theory of Strong Coupling (P. Nozieres)
- Boundary conformal field theory (I. Affleck, A. Ludwig)

- Bosonization (A. Luther, I. Peschel, G. Toulouse)

- Exact solution - (N. Andrei, P. Wiegman)

Renormalization Group Approach

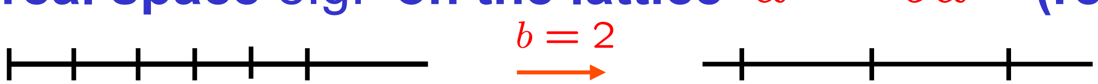
- Why does perturbation theory fail in the IR?
- How to describe Kondo physics at low – *T* ?

The Renormalization Group *(Anderson, Wilson '67-'74)*

Application to the Kondo Problem

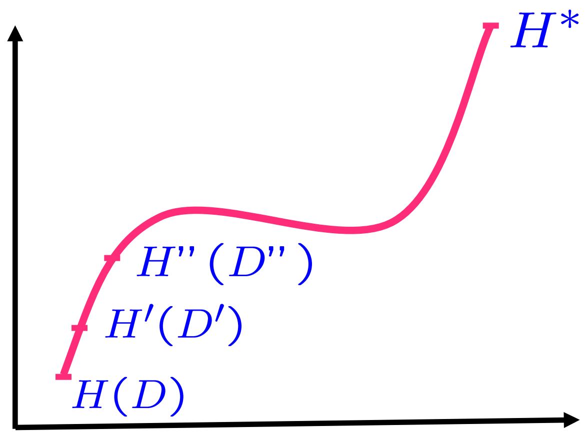
- **Microscopic Hamiltonian written on the scale** D
  - describes physics over the full range of the band-width
- **Construct effective *low–E* Hamiltonian**
  - describes physics close to the Fermi surface (low – *T* physics)
- **Carry out construction step by step ,** $D \to D/b \to D/b^2 \cdots \quad b > 1$
successively integrating out high energy modes, preserving low- *E*

- **In real space** e.g. **on the lattice** $a \longrightarrow ba$ **(recall** $a \sim D^{-1}$ **)**



Renormalization Group Approach

RG flow: the successive Hamiltonians $H, H', H'' \cdots$ describe *same* low-E physics, eliminate high energy modes.



Under RG transformation

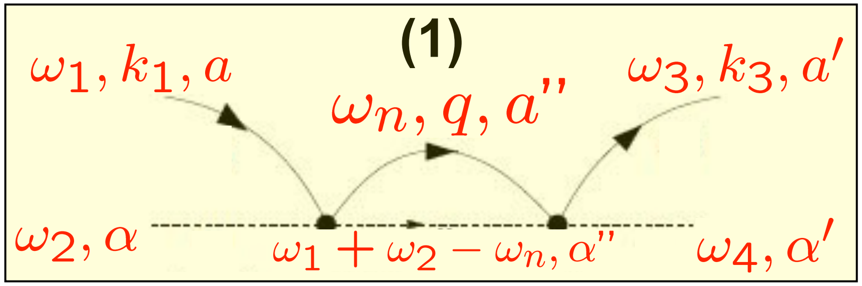
correlation length: $\xi \longrightarrow \xi/b$

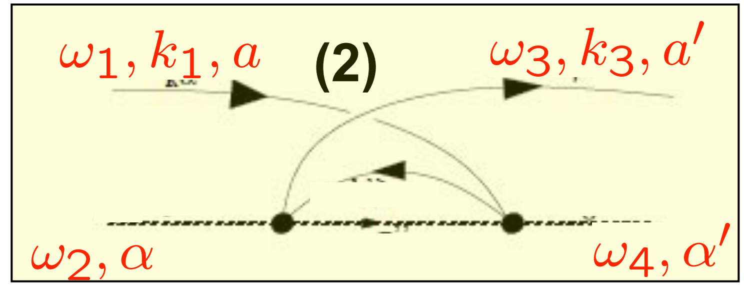
- Coupling constants are running (depend on scale D)
- Is there a fixed point: $H^* \rightarrow H^*$
- Fixed point is scale invariant (conformal invariant) $\xi = \xi/b \longrightarrow \boxed{\xi = \infty, 0}$
- How to construct RG flow?
  - Eliminate (decimate, integrate) high energy modes
  - Rescale
  - Repeat

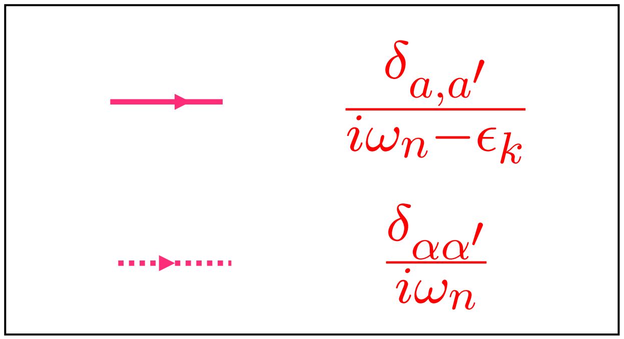
Renormalization Group Approach

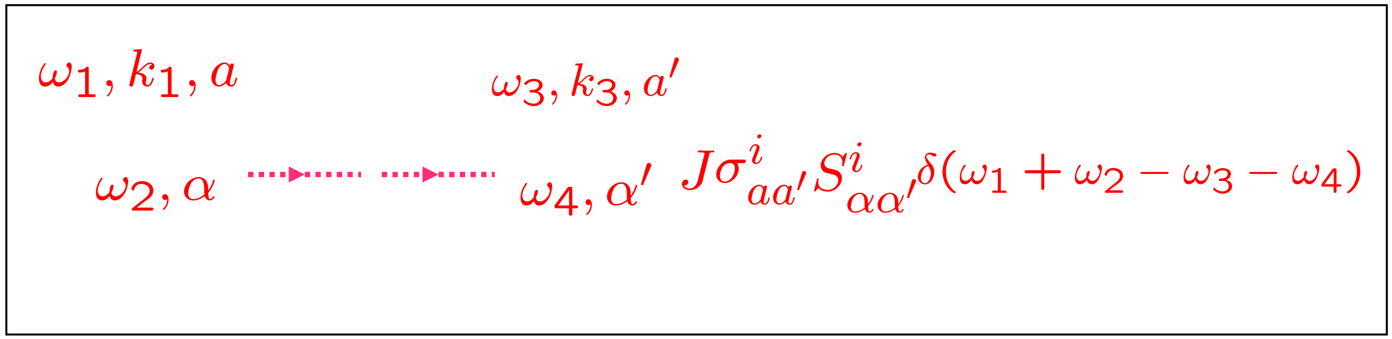
Kondo RG

Contribution of the eliminated modes:









$$
\begin{aligned}
\Delta H_1 &= -\frac{1}{2} J^2 \frac{1}{\beta} \sum_n \int_{D/b}^{D} \frac{d^3 q}{(2\pi)^3} \frac{1}{i\omega_n - \epsilon_q} \frac{1}{i(\omega_1 + \omega_2 - \omega_n)} (\sigma^i \sigma^j)_{aa'} (S^i S^j)_{\alpha,\alpha'} \\
&= -\frac{1}{2} J^2 \rho \frac{1}{\beta} 2 \sum_n \int_{D/b}^{D} d\epsilon \frac{1}{i\omega_n - \epsilon} \frac{1}{i(\omega_1 + \omega_2 - \omega_n)} (\sigma^i \sigma^j)_{aa'} (S^i S^j)_{\alpha,\alpha'} \\
&= -J^2 \rho \frac{1}{\beta} \sum_n \frac{\Delta D}{i\omega_n - D} \frac{1}{i(\omega_1 + \omega_2 - \omega_n)} (\sigma^i \sigma^j)_{aa'} (S^i S^j)_{\alpha,\alpha'} \\
&= -J^2 \rho \frac{\Delta D}{D} (\sigma^i \sigma^j)_{aa'} (S^i S^j)_{\alpha,\alpha'} \\
&= \frac{1}{2} J^2 \rho \frac{\Delta D}{D} (\sigma^i)_{aa'} (S^i)_{\alpha,\alpha'} + \frac{3}{4} J^2 \rho \frac{\Delta D}{D} \delta_{aa'} \delta_{\alpha,\alpha'}
\end{aligned}
$$

modifies Kondo coupling

generates potential scattering (irrelevant)

$$\Delta H_2 = \text{same}$$

Renormalization Group Approach

- **Reduction in band-width compensated by an *increase* in coupling constant**

$$J(D-\Delta D) = J(D) + \frac{\Delta D}{D}\rho J^2 + \cdots$$

e.g. $-\frac{\Delta D}{D}\frac{1}{2}\rho^2 J^3$ (neglect!)

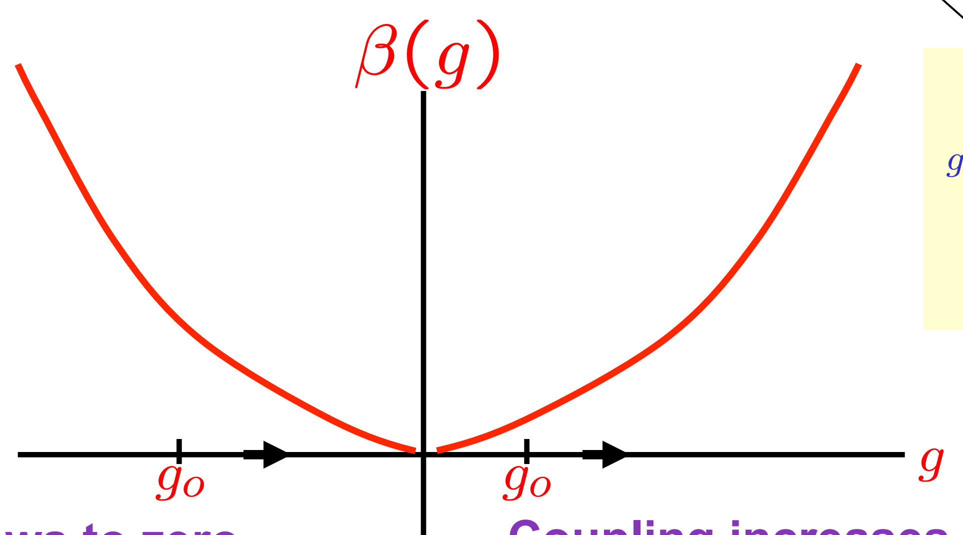
- **Perturbation is IR unstable (*J*>0)**

Denoting $g = \rho J$ we have $\beta(g) \equiv -\frac{\partial g}{\partial \log D} = g^2 > 0$

The running coupling

$$\begin{aligned} g(D) &= \frac{g_0}{1 + g_0 \log(D/D_0)} \\ &= \frac{1}{\log(D/T_K)} \end{aligned}$$

- **Perturbation is IR stable (J<0)**



Coupling flows to zero – perturbation thy IR stable

Coupling increases – where does it flow?

Ferromagnetic regime

Antiferromagnetic regime

Renormalization Group Approach

Where does the coupling flow to?

- Perturbation theory fails when $g(D) \sim 1$
- Nonperturbative approaches:

  - Coulomb gas representation (Andreson, Yuval)
  - Numerical RG (Wilson)
  - Bethe Ansatz (Andrei, Wiegmann)

$$\left.\begin{array}{l}\text{Coulomb gas representation (Andreson, Yuval)}\\ \text{Numerical RG (Wilson)}\\ \text{Bethe Ansatz (Andrei, Wiegmann)}\end{array}\right\} g \longrightarrow \infty$$

- Screening of impurity spin at low –T:
  finite susceptibility ⟷ screened spin

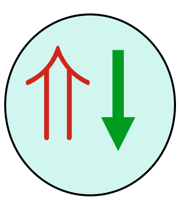
$$\chi^i|_{T=0} = \frac{1}{T_K}\frac{1}{4\pi}\frac{e^{(C+1/4)}}{\sqrt{\pi}} = \frac{1}{T_K}\,0.102676$$

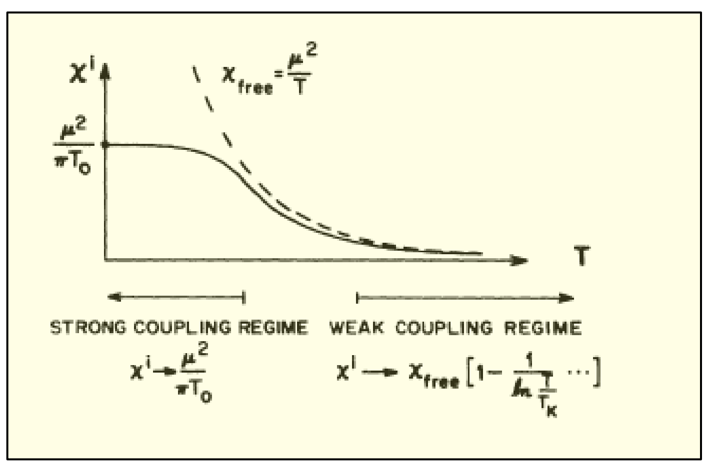
Bethe-Ansatz

0.1032 ±0.0005

Wilson RG

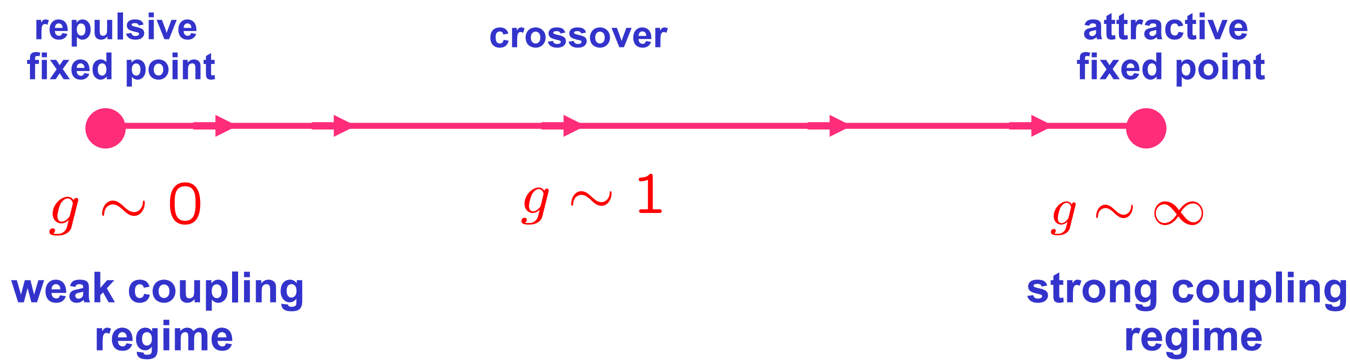
Strongly coupled singlet



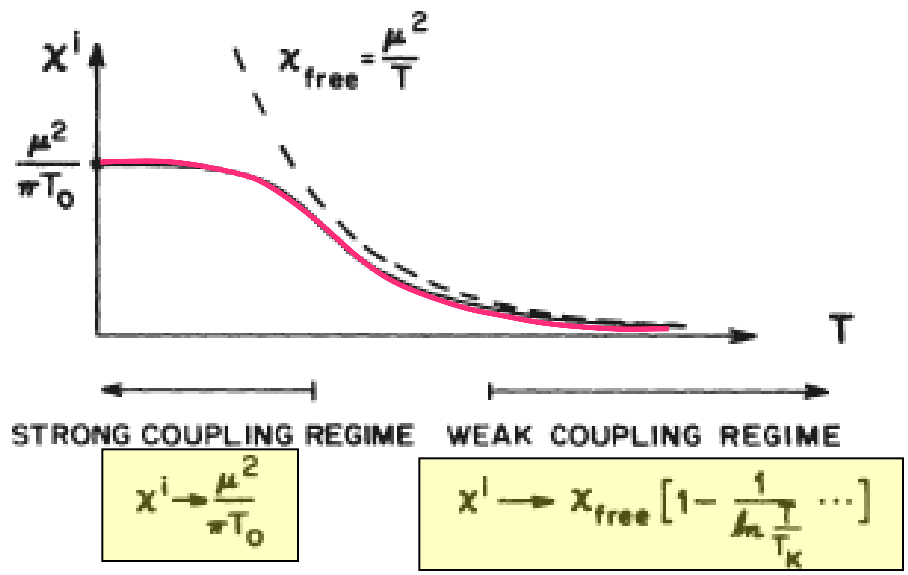


Renormalization Group Approach

- **The RG flow:**



- **The system evolution with temperature:**



- **What is the strong coupling fixed point Hamiltonian**? (Wilson, Nozieres)

The Kondo Problem

Many approaches to the Kondo Problem:

- Resummation of the perturbation series (fails)
- Variational techniques (fail)

- Scaling theory – Poor Man’s Scaling (P. W. Anderson)
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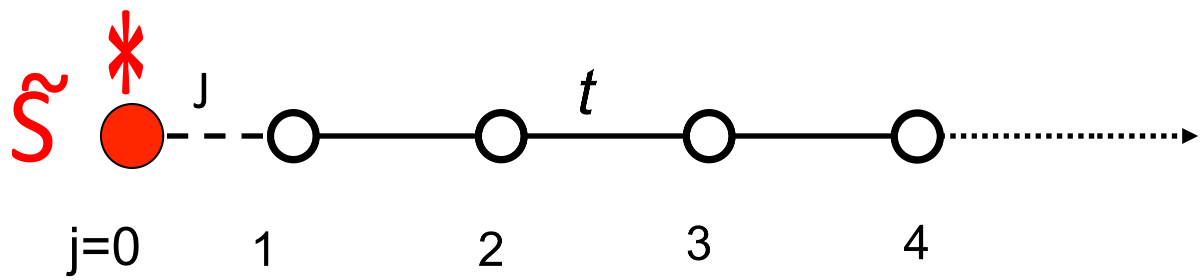
Numerical Renormalization Group Approach

- **How to characterize the strong coupling fixed point Hamiltonian?** (Nozieres)

  - Kondo Hamiltonian on the lattice:

$$H = -t \sum_{j=0}^{\infty} [c_a^\dagger(j+1)c_a(j) + h.c] + Jc_a^\dagger(0)\vec{\sigma}_{aa'}c_{a'}(0)\cdot\vec{S}$$



  - Strong coupling $J \gg t$ ground state: $|gs\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\downarrow\rangle - |\Downarrow\uparrow\rangle)$ **Local singlet**
  - Electron hopping on site-0 breaks singlet, cost $\Delta E = \frac{3}{4}J$

  - Excluding electrons from site $j=0$ corresponds to phase shift $\delta = \frac{\pi}{2}$

    $\sin(k_F x) \to \sin(k_F x - \delta)$ **with** $x = j\alpha$, **so** $\delta = k_F\,\alpha = \frac{\pi}{2\alpha}\alpha = \frac{\pi}{2}$

  - Strong coupling fixed point Hamiltonian (plus leading marginal op) - local FL

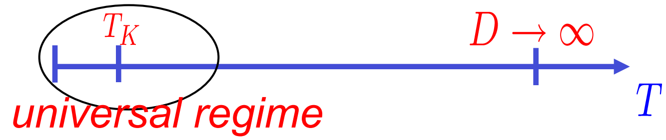
$$H^* = -t \sum_{j=1}^{\infty} [c_a^\dagger(j+1)c_a(j) + h.c] + \frac{t^4}{J^3}n_{1\uparrow}n_{1\downarrow} + \ldots$$

Spin screened, induces interaction among electrons

The Kondo Hamiltonian — field theory

Rewrite the Hamiltonian as 1-dim field theory:

- **Field Theory** *if*: All scales $\ll$ Bandwidth *D*, universal results, independent of band structure
- 1-dim theory

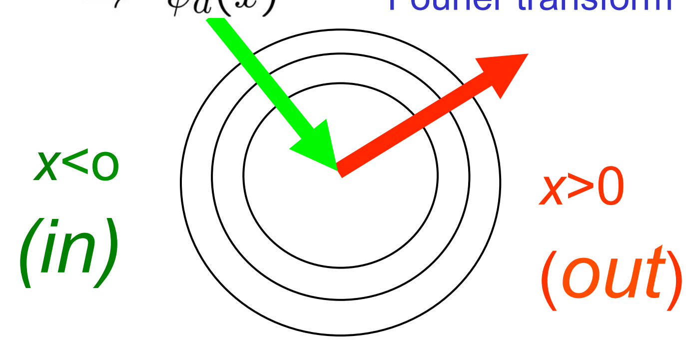


Route 1: sum modes, linearize

$$\begin{aligned}
\psi_{\epsilon a} &\equiv \int d^3k\, \delta(\epsilon_{\vec{k}} - \epsilon)\, c_{\vec{k}a} \\
\{\psi_{\epsilon a}, \psi^\dagger_{j\epsilon' b}\} &= \delta_{ab}\, \delta(\epsilon - \epsilon')\nu(\epsilon) \\
\psi_a(x) &= \int_{-D}^{D} \frac{d\epsilon}{\sqrt{\nu}}\, e^{i\epsilon x}\, \psi_{\epsilon a}
\end{aligned}$$

Route 2: impurity geometry, keep s-waves

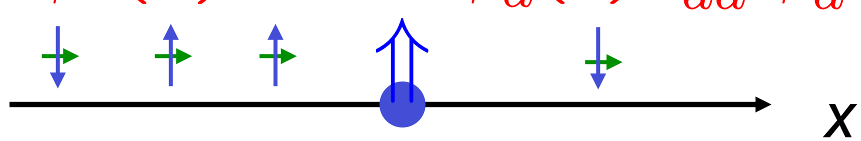
$c_{\vec{k},a}$	$\to$ $c_{k,l,m,a}$	spherical modes	
	$\to$ $c_{k,o,o,a}$	s-waves	
	$\to$ $c_{k_F+q,a}$	linearize around	k_F
	$\to$ $\psi_a(x)$	Fourier transform	



The Kondo Hamiltonian (unfolded):

$$H = -i \int \psi^\dagger_a(x)\partial_x \psi_a(x) dx + J \psi^\dagger_a(o) \vec{\sigma}_{aa'} \psi_{a'}(o) \cdot \vec{S}$$

chiral fermions



$v_F = 1,\ \rho = 1/\pi$

Bethe Ansatz Approach

Steps in the approach

- Construct eigenfunctions of N electrons on ring of length L interacting with the impurity
- Identify ground state, excitations $E_{gs}, E_{spinon}, E_{holon}, E_{quartet}$
- Construct the thermodynamics $Z = \sum_E e^{-\beta E} = e^{-\beta F(T)}$
- Take thermodynamic limit $L \to \infty, N \to \infty, \ D = N/L$ fixed
- Take the scaling limit, universality $D \to \infty, \ T_K$ fixed, then $F(T; D, J) \to F(T/T_K)$
- Compute susceptibility, specific heat, phase shifts..

- Fock space of N electrons spanned by

$$|F\rangle = \int F_{a_1 \ldots a_N; a_0}(x_1 \ldots x_N) \prod_j \psi^\dagger_{a_j}(x_j) |0_F; a_0\rangle$$

- Eigenstate equation $H|F\rangle = E|F\rangle$ becomes: $hF = EF$ with

$$h = -i \sum_{j=1}^{N} \partial_{x_j} + J \sum_{j=1}^{N} \delta(x_j) \vec{\sigma}_j \cdot \vec{S}$$

Kondo Hamiltonian 1-Qu

Bethe Ansatz Approach

Main idea: construct consistently multi-particle wave functions from single-particle wave functions

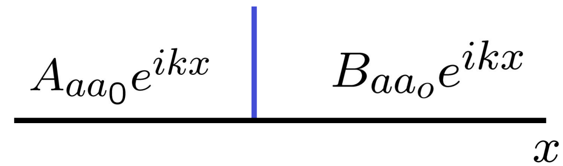
$$h = -i\sum_{j=1}^{N}\partial_{x_j} + J\sum_{j=1}^{N}\delta(x_j)\,\vec{\sigma}_j\cdot\vec{\sigma}_0$$

- For $N=1$, solve

$$[-i\partial_{x_j} + J\delta(x_j)(\vec{\sigma})_{a_j}^{a'_j}\cdot(\vec{\sigma}_o)_{a_o}^{a'_o}]F_{a'_j a'_o}(x_j) = EF_{a_j a_o}(x_j)$$

Solution:

$$F_{a_j a_o}(x_j) = e^{ikx_j}[A_{a_j a_o}\theta(-x_j) + B_{a_j a_o}\theta(x_j)]$$



where: (using $\delta(x)\theta(\pm x) = \frac{1}{2}$)

$$E = k \quad \text{and} \quad -i(A-B) + \tfrac{1}{2}J\vec{\sigma}\cdot\vec{\sigma_o}(A+B) = 0 \quad \longrightarrow B = \frac{i-\frac{1}{2}J\sigma\cdot\sigma_o}{i+\frac{1}{2}J\sigma\cdot\sigma_o}A$$

Thus

$$B_{a_j a_o} = S_{a_j a_o}^{a' a'_o}A_{a'_j a'_o}$$

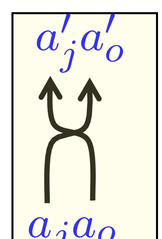
$$F_{a_j a_o}(x_j) = e^{ik_j x}[I_{a_j a_o}^{a'_j a'_o}\theta(-x_j) + S_{a_j a_o}^{a'_j a'_o}\theta(x_j)]A_{a'_j a'_o}$$

S-matrix

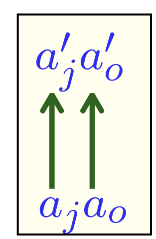
$$S^{jo} \equiv S_{a_j a_o}^{a'_j a'_o} = \frac{I^{jo} - icP^{jo}}{1-ic}$$

$$c = \frac{2J}{1-(3/4)J^2}$$

$$P^{jo} \equiv P_{a_j a'_o}^{a'_j a'_o} = \tfrac{1}{2}[1+\vec{\sigma}_j\cdot\vec{\sigma}_o]_{a_j a'_o}^{a'_j a'_o} = \delta_{a_j}^{a'_o}\delta_{a_o}^{a'_j}$$



$$I^{jo} \equiv I_{a_j a'_o}^{a'_j a'_o} = \delta_{a_j}^{a'_j}\delta_{a_o}^{a'_o}$$



The Yang-Baxter equation

- **Consider $N = 2$ particles**

  - Divide configuration space into $3! = 6$ regions, Q = 1…6, according to ordering,

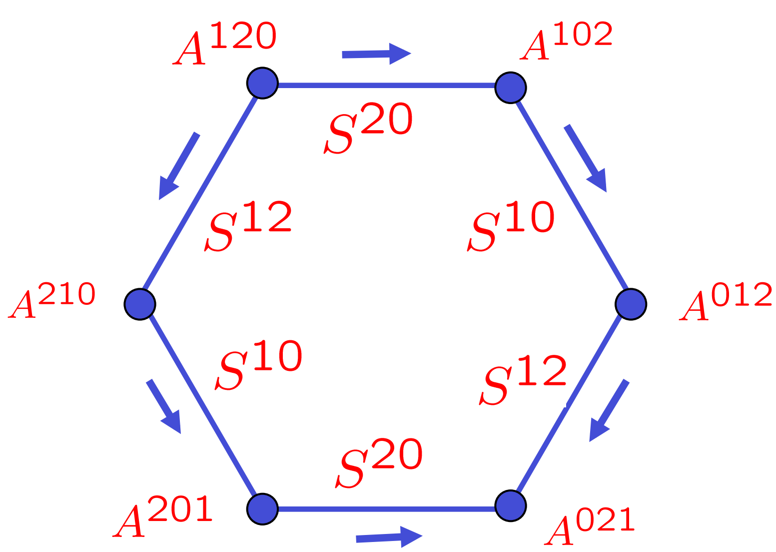
    example: (120) denotes $(x_1 < x_2 < 0)$

  - Inside each region Q the wave function is: $e^{ik_1x_1+k_2x_2}A_{a_1a_2a_0}(Q)$

  - Total wave function $\mathcal{A}\ e^{ik_1x_1+k_2x_2}\sum_Q A_{a_1a_2a_0}(Q)\theta(x_Q)$

  - Regions connected by S-matrices

  - Is the construction consistent?



- Starting from region (120) we can reach region (021) via two paths

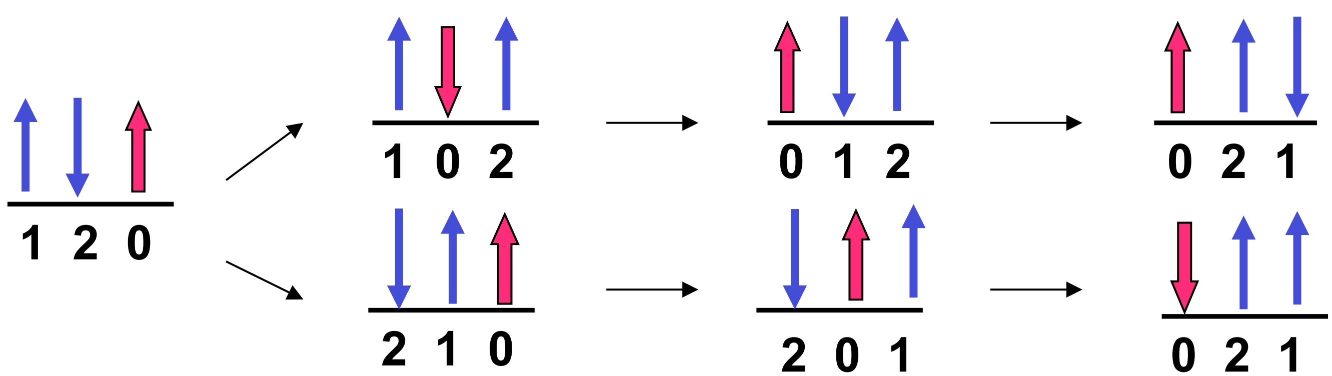
- Construction consistent only if:

$$\boxed{S^{20}\,S^{10}\,S^{12} = S^{12}\,S^{10}\,S^{20}}$$

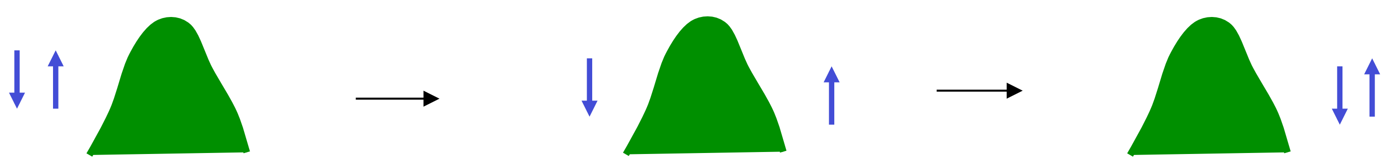
Yang-Baxter equation

On the nature of a quantum impurity

- **Do the S-matrices satisfy YBE?**
  - The electron-impurity S-matrix: $S^{jo} = \frac{I^{jo} - icP^{jo}}{1 - ic}$ derived from Hamiltonian
  - What is the electron-electron S-matrix S^{ij} ?
- **First attempt – electrons do not interact so** $S^{ij} = I^{ij}$
  - YBE $S^{10}\, S^{20} = S^{20}\, S^{10}$ not satisfied
  - Why? Quantum Impurity changes its state when an electron crosses



  - As opposed to a potential which does not change its state



The Bethe basis

• **what** S^{ij} **satisfies YBE? Answer** $S^{ij} = P^{ij}$

• **But have we introduced interactions among electrons?** $h = -i(\partial_{x_1} + \partial_{x_2})$
No! We made a choice of basis of eigenstates for the degenerate subspace corresponding to $E = k_1 + k_2$

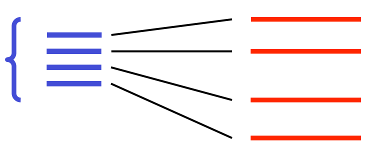
- The linear spectrum $E = k_1 + k_2 = (k_1 + q) + (k_2 - q)$ Infinitely degenarate

- Thus $F = \mathcal{A} e^{i(k_1 x_1 + k_2 x_2)} [A_{a_1 a_2} \theta_{(x_1 - x_2)} + (S\, A)_{a_1 a_2} \theta_{(x_2 - x_1)}]$ eigenfunction for any S

- For $S^{ij} = P^{ij}$ we have charge-spin separation

$$(e^{i(k_1 x_1 + k_2 x_2)} - e^{i(k_1 x_2 + k_2 x_1)})[A_{a_1 a_2} \theta_{(x_1 - x_2)} + A_{a_2 a_1} \theta_{(x_2 - x_1)}]$$

- The choice $S^{ij} = P^{ij}$ defines the **Bethe basis**, the correct basis to turn on interaction from a degenerate level ($S^{ij} = I^{ij}$ is the Fock basis)



degenerate levels

levels split by perturbation

- To perturbe a degenerate level need choose a basis that diagonalizes perturbation – Bethe basis $\frac{|\langle i|H_I|j\rangle|^2}{E_i - E_j}$

- The Bethe basis (unlike Fock basis) *separates* charge and spin since the Kondo interaction is in the spin channel only

Bethe Ansatz Approach

- **For N particles?**
  - The YBE sufficient for all $N = N^e + 1$
  - The consistent wave functions $F = \mathcal{A} e^{\sum_j k_j x_j} \sum_Q A_{a_1..a_{N^e},a_0}(Q)\theta(x_Q)$
    - defined in one reference region $A_{a_1..a_{N^e},a_0}(Q)\theta(x_Q)$(rather than $N!$ regions)
    - same set of $\{k_j\}$ in all regions

Next steps

- **Impose PBC,** $F_{a_1,\ldots,a_N}(x_1,\ldots,x_j = L/2,\ldots,x_N) = F_{a_1,\ldots,a_N}(x_1,\ldots,x_j = -L/2,\ldots,x_N)$
- **Determine spectrum** $E = \sum_j k_j$ **(Bethe Ansatz equations)**
- **Derive free energy and the Thermodynamic Bethe Ansatz eqns (TBA)**
- **The thermodynamic limit and the scaling limit, universality** $\frac{1}{T}F = f(T/T_K, H/T_K)$

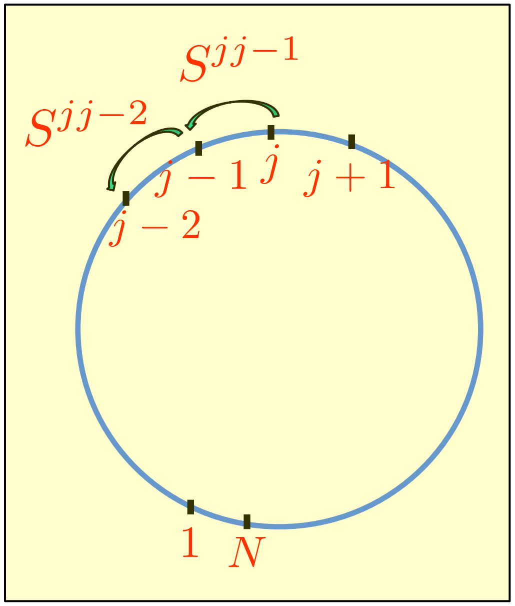
Periodic boundary conditions

- **Impose PBC:**

$$F_{a_1,\ldots,a_N}(x_1,\ldots,x_j = L/2,\ldots,x_N) = F_{a_1,\ldots,a_N}(x_1,\ldots,x_j = -L/2,\ldots,x_N)$$

This translates to the condition

$$(Z_j)^{b_1\ldots b_N}_{a_1\ldots a_N} A_{b_1\ldots b_N}(Q) = e^{-ik_jL} A_{a_1\ldots a_N}(Q)$$



with

$$(Z_j)^{b_1\ldots b_N}_{a_1\ldots a_N} = (S^{jj-1}\ldots S^{j1}S^{jN}\ldots S^{jj+1})^{b_1\ldots b_N}_{a_1\ldots a_N}$$

or in our case

$$(Z_j)^{b_1\ldots b_N}_{a_1\ldots a_N} = \left(P^{jj-1}\ldots P^{j1}P^{jN}\ldots e^{i\phi}\frac{I^{j0} - icP^{j0}}{1-ic}\ldots P^{jj+1}\right)^{b_1\ldots b_N}_{a_1\ldots a_N}$$

The eigenvalues of the Z_j- matrices yield the momenta k_j from which the spectrum can be determined: $E = \sum k_j$

- **How to diagonalize** Z_j?

Algebraic Bethe Ansatz approach

1. **Define:** S-matrix depending on a continuous variable (spectral parameter)

$$S(\alpha) = \frac{\alpha I - icP}{\alpha - ic} \equiv a(\alpha)I + b(\alpha)P$$

assign $\alpha = 1$ to an electron and $\alpha = 0$ to the impurity (α corresponds to the velocity)

we have: el-imp $S^{j0}(\alpha_j - \alpha_0) = S^{j0}(1) = \frac{1 - icP^{j0}}{1 - ic}$ and el-el $S^{jl}(\alpha_j - \alpha_l) = S^{j0}(0) = P^{jl}$

- The S-matrices satisfy a continuous YBE,

$$S^{kj}(\alpha - \beta)S^{ki}(\alpha)S^{ji}(\beta) = S^{ji}(\beta)S^{ki}(\alpha)S^{kj}(\alpha - \beta)$$

- Each electron and impurity has a spin space $\mathcal{C}^2$ associated with it.

  Define and auxiliary spin space A and S-matrices S^{jA}, S^{0A}

2. **Define:** Monodromy matrix

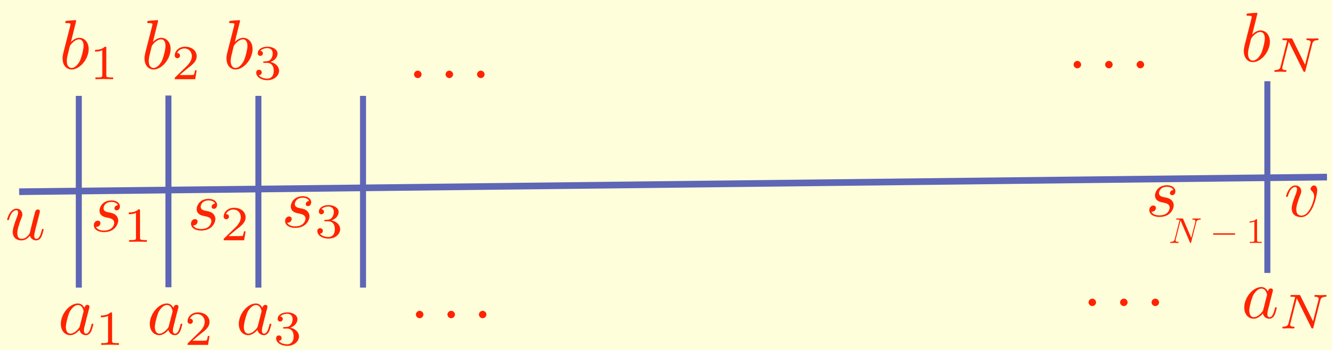
$$\mathcal{M}(\alpha) = S^{1A}(\alpha - \alpha_1)S^{2A}(\alpha - \alpha_2)......S^{NA}(\alpha - \alpha_N)$$

Algebraic Bethe Ansatz Approach

Explicitely:

$$[\mathcal{M}(\alpha)]^{b_1\ldots b_N,v}_{a_1\ldots a_N,u} = \sum_{s_1\ldots s_{N-1}} [S^{1A}(\alpha-\alpha_1)]^{b_1,s_1}_{a_1,u}[S^{2A}(\alpha-\alpha_2)]^{b_2,s_2}_{a2,s_1}\ldots\ldots[S^{NA}(\alpha-\alpha_N)]^{b_N,v}_{a_N,s_{N-1}}$$

$$(\mathcal{M})^{b_1\ldots b_N,v}_{a_1\ldots a_N,u} = \sum_{s_1\ldots s_{N-1}}$$



Represent monodromy matrix in the auxiliary space:

$$[\mathcal{M}(\alpha)]^{b_1\ldots b_N}_{a_1\ldots a_N} = \left[\begin{array}{cc} A^{b_1\ldots b_N}_{a_1\ldots a_N}(\alpha) & B^{b_1\ldots b_N}_{a_1\ldots a_N}(\alpha) \\ C^{b_1\ldots b_N}_{a_1\ldots a_N}(\alpha) & D^{b_1\ldots b_N}_{a_1\ldots a_N}(\alpha) \end{array}\right]$$

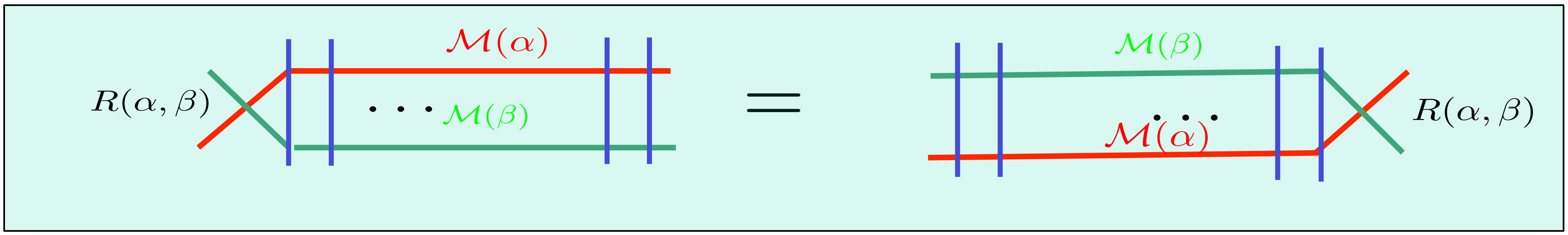
Algebraic Bethe Ansatz Approach

3. The monodromy matrices satisfy:

$$R(\alpha-\beta)\ (\mathcal{M}(\alpha)\otimes\mathcal{M}(\beta)) = (\mathcal{M}(\beta)\otimes\mathcal{M}(\alpha))\ R(\alpha-\beta)$$

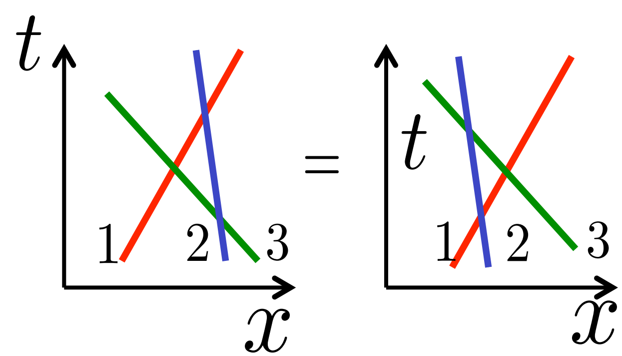
with $R = S(\alpha-\beta)\ P = \dfrac{(\alpha-\beta)P + icI}{(\alpha-\beta)+ic}$ explicitly: $R(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{ic}{\alpha+ic} & \frac{\alpha}{\alpha+ic} & 0 \\ 0 & \frac{\alpha}{\alpha+ic} & \frac{ic}{\alpha+ic} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

proof



- follows from YBE

$$S^{23}S^{13}S^{12} = S^{12}S^{13}S^{23}$$



Algebraic Bethe Ansatz Approach

4. The commutation relations:

We saw

$$R(\alpha-\beta)\ (\mathcal{M}(\alpha)\otimes\mathcal{M}(\beta)) = (\mathcal{M}(\beta)\otimes\mathcal{M}(\alpha))\ R(\alpha-\beta)$$

with

$$R(\alpha-\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{ic}{\alpha-\beta+ic} & \frac{\alpha-\beta}{\alpha-\beta+ic} & 0 \\ 0 & \frac{\alpha}{\alpha-\beta+ic} & \frac{ic}{\alpha-\beta+ic} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$[\mathcal{M}(\alpha)] = \begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix}$$

Hence

$$A(\alpha)B(\beta) = u(\beta-\alpha)B(\beta)A(\alpha) + v(\beta-\alpha)B(\alpha)A(\beta)$$
$$D(\alpha)B(\beta) = u(\alpha-\beta)B(\beta)D(\alpha) + v(\alpha-\beta)B(\alpha)D(\beta)$$
$$A(\alpha)A(\beta) = A(\beta)A(\alpha)$$
$$B(\alpha)B(\beta) = B(\beta)B(\alpha)$$
$$D(\alpha)D(\beta) = D(\beta)D(\alpha)$$

$$\boxed{v(\alpha) = -\frac{ic}{\alpha} \qquad u(\alpha) = \frac{\alpha+ic}{\alpha}}$$

Algebraic Bethe Ansatz Approach

5. **Define:** Transfer matrix $\mathcal{T}(\alpha) = \mathrm{Tr}_{\mathrm{A}}\, \mathcal{M}(\alpha)$

Explicitly: $[\mathcal{T}(\alpha)]^{b_1\ldots b_N}_{a_1\ldots a_N} = \sum_u [\mathcal{M}(\alpha)]^{b_1\ldots b_N,u}_{a_1\ldots a_N,u} = [A(\alpha)]^{b_1\ldots b_N}_{a_1\ldots a_N} + [B(\alpha)]^{b_1\ldots b_N}_{a_1\ldots a_N}$

Claim: $[\mathcal{T}(\alpha), \mathcal{T}(\beta)] = 0 \qquad \forall\, \alpha, \beta$

Recall, we wish to diagonalize $Z_j = S^{jj-1} \ldots S^{j1} S^{jN} \ldots S^{jj+1}$

Claim: $Z_j = \mathcal{T}(\alpha_j) = A(\alpha_j) + D(\alpha_j)$

6. **Claim:** $B(\beta)$ - creation operator w.r.t Hamiltonian $A(\alpha) + D(\alpha)$

when acting on ferromagnetic vacuum $|\omega >= \prod_{j=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j$, up to *unwanted terms:*

Write:

$$S^{jA}(\alpha) = (a + \frac{1}{2}b)(\alpha) 1_j 1_A + \frac{1}{2} b(\alpha) \sigma_j \cdot \sigma_A = \begin{pmatrix} (a+\frac{1}{2}b)(\alpha)1_j + \frac{1}{2}b(\alpha)\sigma^z_j & b(\alpha)\sigma^-_j \\ b(\alpha)\sigma^+_j & (a+\frac{1}{2}b)(\alpha)1_j - \frac{1}{2}b(\alpha)\sigma^z_j \end{pmatrix}$$

so $|\omega\rangle$ is eigenstate of $A(\alpha)$ and of $D(\alpha)$: $A(\alpha)|\omega >= |\omega >$

$$D(\alpha)|\omega >= \prod_{j=1}^{N} \frac{\alpha - \alpha_j}{\alpha - \alpha_j + ic} |\omega >$$

Algebraic Bethe Ansatz Approach

Consider now the state (with M flipped spin):

$$|F(\beta_1...\beta_M)> \equiv B(\beta_1)...B(\beta_M)|\omega> = \sum_{j_1...j_M} F_{j_1...j_M}\sigma^-_{j_1}...\sigma^-_{j_M}|\omega>$$

Applying the Hamiltonian $A(\alpha)+D(\alpha)$ we find:

$$(A(\alpha)+D(\alpha))B(\beta_1)B(\beta_2)|\omega> =$$

$$u(\beta_1-\alpha)u(\beta_2-\alpha)B(\beta_1)B(\beta_2)A(\alpha)|\omega> +u(\alpha-\beta_1)u(\alpha-\beta_2)B(\beta_1)B(\beta_2)D(\alpha)|\omega>$$

$$+[u(\beta_1-\alpha)v(\beta_2-\alpha)+v(\beta_1-\alpha)v(\beta_2-\beta_1)]B(\alpha)B(\beta_1)A(\beta_2)|\omega>$$

$$+[u(\alpha-\beta_1)v(\alpha-\beta_2)+v(\alpha-\beta_1)v(\beta_1-\beta_2)]B(\alpha)B(\beta_1)D(\beta_2)|\omega>$$

$$+v(\beta_1-\alpha)u(\beta_2-\beta_1)B(\alpha)B(\beta_2)A(\beta_1)|\omega> +v(\alpha-\beta_1)u(\beta_1-\beta_2)B(\alpha)B(\beta_2)D(\beta_1)|\omega>$$

$$= \lambda(\alpha,\beta_1\beta_2)B(\beta_1)B(\beta_2)|\omega> +\lambda_1(\alpha,\beta_1\beta_2)B(\alpha)B(\beta_2)|\omega> +\lambda_2(\alpha,\beta_1\beta_2)B(\alpha)B(\beta_1)|\omega>$$

with

$$\lambda(\alpha,\beta_1\beta_2) = u(\beta_1-\alpha)u(\beta_2-\alpha) + \prod_{j=1}^{N}\frac{\alpha-\alpha_j}{\alpha-\alpha_j+ic}u(\alpha-\beta_1)u(\alpha-\beta_2)$$

The eigenvalue

$$\left.\begin{aligned}\lambda_1(\alpha,\beta_1\beta_2) &= v(\beta_1-\alpha)[u(\beta_2-\beta_1)-u(\beta_1-\beta_2)\prod_{j=1}^{N}\frac{\beta_1-\alpha_j}{\beta_1-\alpha_j+ic}] \\ \lambda_2(\alpha,\beta_1\beta_2) &= v(\beta_2-\alpha)[u(\beta_1-\beta_2)-u(\beta_2-\beta_1)\prod_{j=1}^{N}\frac{\beta_2-\alpha_j}{\beta_2-\alpha_j+ic}]\end{aligned}\right\}$$

Unwanted terms
– set to zero

$$\lambda_\gamma(\alpha,\beta_1\beta_2)=0,\ \gamma=1,2$$

Algebraic Bethe Ansatz Approach

7. Recall we want eigenvalues of $Z_j = Z(\alpha = \alpha_j)$

We showed:

$$z_j = \lambda(\alpha_j, \beta_1 \dots \beta_M) = \prod_{\gamma=1}^{M} \frac{\beta_\gamma - \alpha_j + ic}{\beta_\gamma - \alpha_j}$$

provided that $\beta_1, .., \beta_M$ satisfy the Bethe Ansatz equations (BAE):

$$\prod_{\delta=1, \delta\neq\gamma}^{M} \frac{\beta_\delta - \beta_\gamma + ic}{\beta_\delta - \beta_\gamma - ic} = \prod_{i=1}^{N} \frac{\beta_\gamma - \alpha_i}{\beta_\gamma - \alpha_i + ic}$$

8. Setting $\beta_\gamma = \Lambda_\gamma - ic/2$ and recalling $z_j = e^{-ik_j L}$ we have:

$$e^{ik_j L} = \prod_{\gamma=1}^{M} \frac{\Lambda_\gamma - \alpha_j - ic/2}{\Lambda_\gamma - \alpha_j + ic/2}$$

$$\prod_{\delta=1, \delta\neq\gamma}^{M} \frac{\Lambda_\delta - \Lambda_\gamma + ic}{\Lambda_\delta - \Lambda_\gamma - ic} = \prod_{i=1}^{N} \frac{\Lambda_\gamma - \alpha_i - ic/2}{\Lambda_\gamma - \alpha_i + ic/2}$$

with $\Lambda_\gamma, \ \gamma = 1, .., M$ solutions of the BAE

For Kondo: $\alpha_j = 0, 1$

For Hubbard $\alpha_j = \sin k_j$

For Yang $\alpha_j = k_j$

The Solution

A system of N^e electrons interacting with the Kondo impurity -

- The momenta for state with M spins down, $N^e + 1 - M$ spins up:

$$k_j = \frac{2\pi}{L} n_j + \frac{1}{L} \sum_{\gamma=1}^{M} [\Theta(2\Lambda_\gamma - 2) - \pi], \qquad \text{with} \quad \Theta(x) = -2\tan^{-1}(x/c)$$

- The energy:

$$\boxed{E = \sum_{j=1}^{N^e} k_j \ = \sum_{j=1}^{N^e} \frac{2\pi}{L} n_j + D \sum_{\gamma=1}^{M} [\Theta(2\Lambda_\gamma - 2) - \pi],}$$

- with the spin momenta Λ_γ satisfying:

$$-\prod_{\delta=1}^{M} \frac{\Lambda_\delta - \Lambda_\gamma + ic}{\Lambda_\delta - \Lambda_\gamma - ic} = \left(\frac{\Lambda_\gamma - 1 - ic/2}{\Lambda_\gamma - 1 + ic/2} \right)^{N^e} \left(\frac{\Lambda_\gamma - ic/2}{\Lambda_\gamma + ic/2} \right)$$

or

$$\boxed{N^e \Theta(2\Lambda_\gamma - 2) + \Theta(2\Lambda_\gamma) = -2\pi I_\gamma + \sum_{\delta=1}^{M} \Theta(\Lambda_\gamma - \Lambda_\delta), \qquad \gamma = 1....M}$$

- The spin of the system is: $S = S_z = \frac{1}{2}(N^e + 1) - M$

Extracting the physics

How to extract the physics from these equations (in five easy steps)**?**

- Identify ground state, excitations $E_{gs}, E_{spinon}, E_{holon}, E_{quartet}$
- Construct the thermodynamics $Z = \sum_E e^{-\beta E} = e^{-\beta F(T)}$
- Take thermodynamic limit $L \to \infty, N \to \infty, \; D = N/L$ fixed
- Take the scaling limit, universality $D \to \infty, \; J \to 0; \;\; T_K = De^{-1/J}$ fixed,

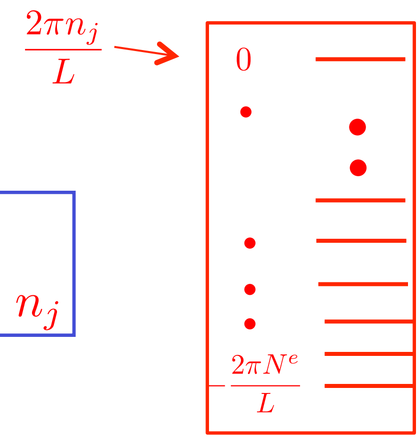
  - the free energy takes the scaling form $F = T\, f(T/T_K, h/T)$

- Compute susceptibility, specific heat, phase shifts..

  - Study high- and low- temperature behavior and the crossover between them

Eigenstates

BAE for an eigenstate with M spins down and $N^e + 1 - M$ spins up,



Charge Fermi sea

$$E = \sum_{j=1}^{N^e} \frac{2\pi}{L} \boxed{n_j} + D \sum_{\gamma=1}^{M} [\Theta(2\Lambda_\gamma - 2) - \pi],$$

Note: linear spectrum requires a cut-off, $-N^e \leq n_j$

$$N^e \Theta(2\Lambda_\gamma - 2) + \Theta(2\Lambda_\gamma) = -2\pi \boxed{I_\gamma} + \sum_{\delta=1}^{M} \Theta(\Lambda_\gamma - \Lambda_\delta), \qquad \gamma = 1....M$$

- The integers $\{n_j, I_\gamma\}$ are the quantum numbers of the eigenstates

  $-(N^e - M)/2 \leq I_\gamma \leq (N^e - M)/2$ and $-N^e \leq n_j$ so $D = N^e/L$ is the *cut-off*

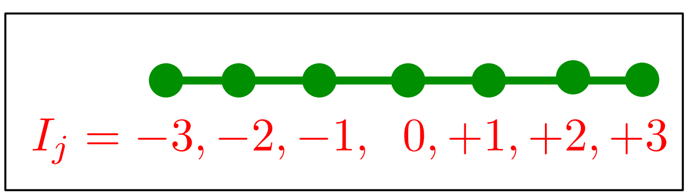
-There is charge spin separation: $\{n_j\}$ determine charge-dynamics, charge Fermi sea

$\{I_\gamma\}$ Determine spin-dynamics, spin Fermi sea

- Ground state configuration $\{n_j^{gs}, I_\gamma^{gs}\}$ is a singlet $M = (N^e + 1)/2$ with:

$I_{\gamma+1} = I_\gamma + 1$, occupy all slots $-(N^e - M)/2 \leq I_\gamma \leq (N^e - M)/2$

e.g. $N^e = 13$

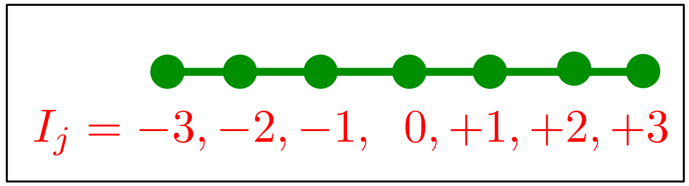


Ground state and excitations

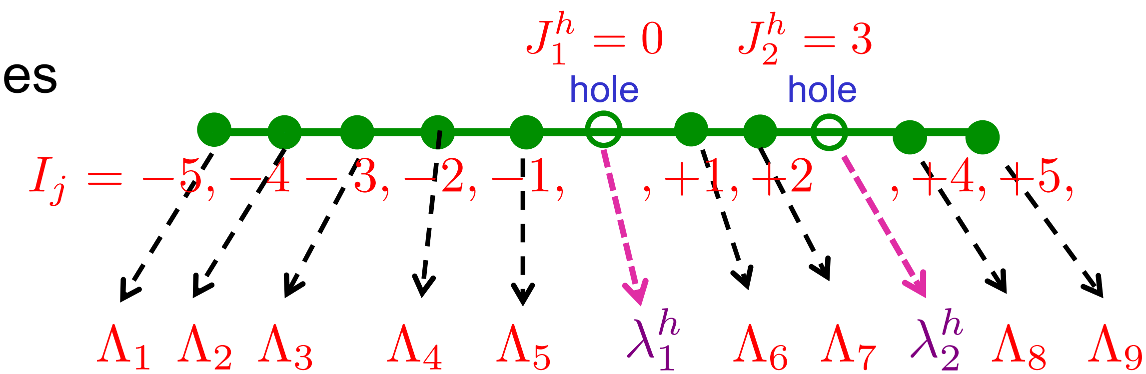
- Ground state configuration $\{n_j^{gs}, I_\gamma^{gs}\}$: with $M = (N^e + 1)/2$ corresponds to a singlet

$I_{\gamma+1} = I_\gamma + 1$, occupy all slots $-(N^e - M)/2 \le I_\gamma \le (N^e - M)/2$

e.g. $N^e = 13$



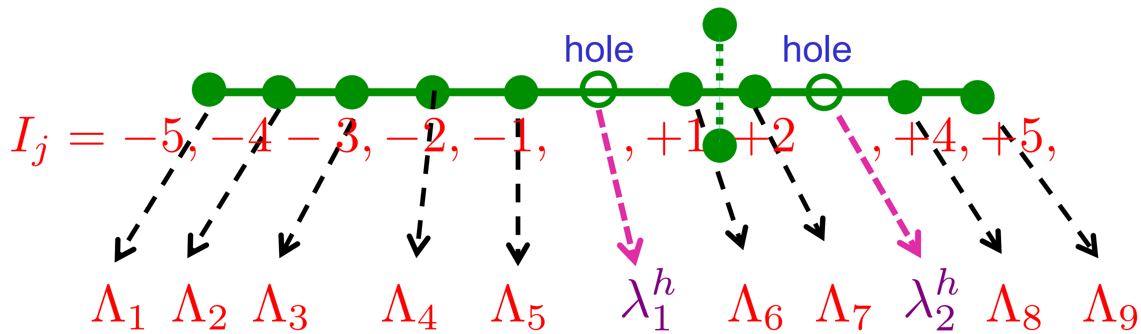
Excitations: holes



Skipped integers J_l^h are holes in the I_j sequence – **excitations**

strings

e.g. 2-string $\lambda^\pm = (\lambda_1^h + \lambda_2^h)/2$



The ground state

In the thermodynamic limit - interested in the density of solutions: $\sigma(\Lambda)$

$\sigma(\Lambda)d\Lambda$ number of solutions in $d\Lambda$, equivalently: $\sigma(\Lambda_\gamma) = 1/(\Lambda_{\gamma+1} - \Lambda_\gamma)$

- Turn BAE into integral equations for $\sigma(\Lambda)$:

Consider the ground state configuration - $I_{\gamma+1} = I_\gamma + 1$

subtract the eqn for Λ_γ

$$N^e\Theta(2\Lambda_\gamma - 2) + \Theta(2\Lambda_\gamma) = -2\pi I_\gamma + \sum_{\delta=1}^{M} \Theta(\Lambda_\gamma - \Lambda_\delta)$$

from the eqn for $\Lambda_{\gamma+1}$

$$N^e\Theta(2\Lambda_{\gamma+1} - 2) + \Theta(2\Lambda_{\gamma+1}) = -2\pi I_{\gamma+1} + \sum_{\delta=1}^{M} \Theta(\Lambda_{\gamma+1} - \Lambda_\delta)$$

and expand in $\Lambda_{\gamma+1} - \Lambda_\gamma \sim 1/N^e$

$$\rightarrow \quad \boxed{\sigma_{gs}(\Lambda) = f(\Lambda) - \int K(\Lambda - \Lambda')\sigma_{gs}(\Lambda')d\Lambda'}$$

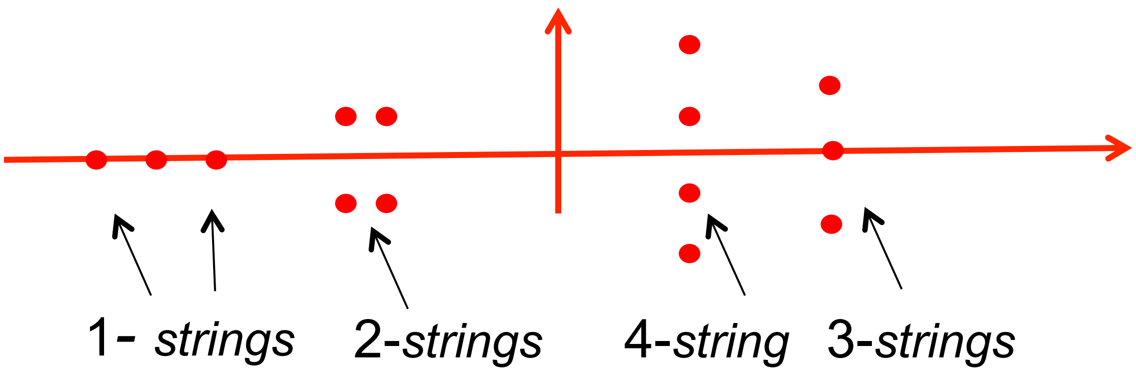
with $K(\Lambda) = \frac{1}{\pi}\frac{c}{c^2 + \Lambda^2}$ and $f(\Lambda) = \frac{2c}{\pi}\left[\frac{N^e}{c^2 + 4(\Lambda - 1)^2} + \frac{1}{c^2 + 4\Lambda^2}\right]$

- Similarly, for each state $\{I_\gamma\}$ determine the corresponding density $\sigma_{\{I_\gamma\}}(\Lambda)$

String solutions

Solutions of the BAE take the form of *n*-strings:

$$\Lambda_j^{(n)} = \Lambda^{(n)} + i\frac{c}{2}(n+1-2j), \quad j = 1,2,\ldots,n.$$



The contribution of a *n*-string to energy

$$D\sum_{j=1}^{n}\left[\Theta(2\Lambda_j^{(n)}-2)-\pi\right] = D\left[\Theta_n(\Lambda^{(n)}-1)-\pi\right]$$

with $\Theta_n(x) = \Theta\big(2x/n\big)$

recall: $E = \sum_{j=1}^{N^e}\frac{2\pi}{L}n_j + D\sum_{\gamma=1}^{M}[\Theta(2\Lambda_\gamma-2)-\pi],$

- They are determined by quantum numbers $I_\gamma^{(n)}$ which induce densities of solutions $\sigma_n(\Lambda)$ and of densities of holes $\sigma_n^h(\Lambda)$

- The densities satisfy the BAE:

$$f_n(\Lambda) = \sigma_n^h(\Lambda) + \sum_{m=1}^{\infty} A_{nm}\sigma_m(\Lambda)$$

where:

$$f_n(\Lambda) = N^e K_n(\Lambda-1) + K_n(\Lambda) \qquad \text{with} \qquad K_n(x) = \frac{1}{\pi}\,\frac{n\frac{c}{2}}{(n\frac{c}{2})^2+x^2}$$

$$A_{nm} = [|n-m|] + 2\,[|n-m|+2] + \cdots + 2\,[n+m-2] + [n+m]$$

$$[n]f(\Lambda) = \int K_n(\Lambda-\Lambda')f(\Lambda')d\Lambda'.$$

The thermodynamics

The thermodynamics of the Kondo model

- The energy $E = E^{(c)}(\{n_j\}) + E^{(s)}(\{I_\gamma^{(n)}\})$

contribution of *n*-string to energy

$$D\sum_{j=1}^{n}\left[\Theta(2\Lambda_j^{(n)}-2)-\pi\right] = D\left[\Theta_n(\Lambda^{(n)}-1)-\pi\right]$$

with $\Theta_n(x) = \Theta(2x/n)$

Charge energy $E^{(c)}(\{n\}) = \dfrac{2\pi}{L}\displaystyle\sum_{j=1}^{N^e} n_j$

Spin energy $E^{(s)}(\{I_\gamma^{n}\}) = D\displaystyle\sum_{\gamma=1}^{M}\Big[\Theta(2\Lambda_\gamma-2)-\pi\Big] = D\sum_n\int d\Lambda\sigma_n(\Lambda)\left[\Theta_n(\Lambda-1)-\pi\right]$

- The partition function $Z = \displaystyle\sum_E \exp\left[-\frac{1}{T}(E-2\mu h S_z)\right] = Z^{(c)}Z^{(s)}$

The charge partition function describes the thermodynamics of N^e non-interacting spinless fermions with linear kinetic energy

$$Z^{(c)} = \sum_{\{n_j\},n_j\geq -N^e}\exp\left[-\frac{1}{T}\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j\right] = e^{-F^{(c)}/T}$$

and taking the cut-off to infinity $D = \dfrac{N^e}{L} \to \infty$ we have

$$F^{(c)} = -\frac{LT}{2\pi}\int_{-\infty}^{\infty} dk\ln\left(1+e^{-\frac{k}{T}}\right) = -\frac{\pi}{12}LT^2 + \{\text{infinite constant}\}$$

Thermodynamics, Yang-Yang entropy

The spin partition function - Sum over all solutions of BAE induced by configurations $\{I_\gamma^{(n)}\}$

$$Z^{(s)} = \sum_{\{I_\gamma^{(n)}\}} \exp\left[-\frac{1}{T}[E^{(s)}(I_\gamma^{(n)}) + 2M\mu h]\right] = \sum_M \sum_{\{\Lambda_1,\ldots,\Lambda_M\}} \exp\left[-\frac{1}{T}[E^{(s)}(\{\Lambda\}) + 2M\mu h]\right]$$

Rewrite in terms of string densities:

$$E^{(s)}(\{\Lambda\}) + 2\mu h M = \sum_n \int d\Lambda \sigma_n(\Lambda) g_n(\Lambda)$$ with $$g_n(\Lambda) = D\big[\Theta_n(\Lambda - 1) - \pi\big] + 2\mu h n$$

$$\boxed{Z^{(s)} = \int \prod D\sigma_n D\sigma_n^h \exp \mathcal{S}(\{\sigma_n, \sigma_n^h\})\ \exp[-\frac{1}{T}\sum_n \int d\Lambda \sigma_n(\Lambda) g_n(\Lambda)]|_{\sigma' s\ -\text{solutions of BAE}}}$$

The replacement of summation over microstates $\{I_\gamma^{(n)}\}$ by summation over densities $\{\sigma_n, \sigma_n^h\}$ requires the introduction of the Yang-Yang entropy $\mathcal{S}(\{\sigma_n, \sigma_n^h\})$ which counts the number of microstates which yield the same densities.

Claim:

$$\mathcal{S}(\{\sigma_n, \sigma_n^h\}) = \sum_n \int d\Lambda[(\sigma_n + \sigma_n^h)\ln(\sigma_n + \sigma_n^h) - \sigma_n^h \ln \sigma_n^h - \sigma_n \ln \sigma_n].$$

The free energy

- The number of slots for n-strings in the interval $d\Lambda$ is, $(\sigma_n + \sigma_n^h) d\Lambda$ of which $\sigma_n d\Lambda$ are occupied, while $\sigma_n^h d\Lambda$ are empty; thus the number of ways of distributing the n-strings among the slots is:

$$\frac{[(\sigma_n(\Lambda) + \sigma_n^h(\Lambda)) d\Lambda]!}{[\sigma_n(\Lambda) d\Lambda]! [\sigma_n^h(\Lambda) d\Lambda]!}$$

Using Stirling's formula, we can simplify this to give

$$d\mathcal{S}_n = \ln \frac{[(\sigma_n + \sigma_n^h) d\Lambda]!}{[\sigma_n d\Lambda]! [\sigma_n^h d\Lambda]!} = [(\sigma_n + \sigma_n^h) \ln(\sigma_n + \sigma_n^h) - \sigma_n \ln \sigma_n - \sigma_n^h \ln \sigma_n^h] d\Lambda$$

- In thermodynamic limit, $N^e \to \infty$, we may evaluate $Z^{(s)}$ by the method of stationary phase approximation, varying the functional,

$$F^{(s)}\{\sigma_n, \sigma_n^h\} = E^{(s)} + 2\mu h M - T\mathcal{S} = \sum_n \int d\Lambda \left[\sigma_n g_n - T\sigma_n \ln\left[1 + \frac{\sigma_n^h}{\sigma_n}\right] - T\sigma_n^h \ln\left[1 + \frac{\sigma_n}{\sigma_n^h}\right]\right]$$

subject to the constraint (the BAE)

$$\delta\sigma_n^h = -\sum_m A_{nm} \delta\sigma_m$$

we obtain the TBA (thermodynamic BA) eqns:

TBA eqns

The TBA eqns:

$$\ln \eta_1 = -\frac{2D}{T} \tan^{-1} e^{(\pi/c)(\Lambda-1)} + G \ln(1+\eta_2)$$

$$\ln \eta_n = G[\ln(1+\eta_{n+1}) + \ln(1+\eta_{n-1})]$$

denote

$$\eta_n(\Lambda) = \frac{\sigma_n^h(\Lambda)}{\sigma_n(\Lambda)}$$

where $$Gf(\Lambda) = \frac{1}{2c}\int d\Lambda' \frac{1}{\cosh \frac{\pi}{c}(\Lambda-\Lambda')} f(\Lambda')$$

and $$\lim_{n\to\infty}([n+1]\ln(1+\eta_n) - [n]\ln(1+\eta_{n+1})) = -\frac{2\mu h}{T}$$

Once the $\{\eta_n(\Lambda)\}$ are determined, the spin free energy is:

$$F^{(s)} = -T\int d\Lambda \frac{1}{2c}\left[\frac{N^e}{\cosh \frac{\pi}{c}(\Lambda-1)} + \frac{1}{\cosh \frac{\pi}{c}\Lambda}\right] \ln\left(1+\eta_1(\Lambda)\right)$$

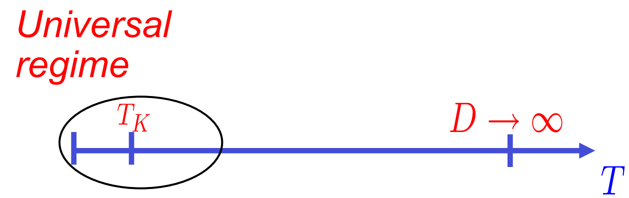
Free energy the spin sector of free gas of electrons

Impurity free energy $F^{(imp)}$

Scaling of the thermodynamics

- Scaling properties of the TBA eqns (in the universal regime):

$$\ln\eta_1 = -\frac{2D}{T}\tan^{-1} e^{(\pi/c)(\Lambda-1)} + G\ln(1+\eta_2)$$
$$\ln\eta_n = G[\,\ln(1+\eta_{n+1}) + \ln(1+\eta_{n-1})]$$



In this regime $\eta_1 \sim \exp[-(2D/T)\tan^{-1} z]$ *has contribution only for* $z = \exp[(\pi/c)(\Lambda-1)] \ll 1$

Thus: $$\frac{2D}{T}\tan^{-1} e^{(\pi/c)(\Lambda-1)} \to \frac{2D}{T} e^{(\pi/c)(\Lambda-1)} \to \frac{2T_K}{T} e^{(\pi/c)\Lambda} \to 2e^{\zeta}$$

with $\boxed{T_K = De^{-\pi/c}}$ and $\zeta = \frac{\pi}{c}\Lambda + \ln\frac{T_K}{T}$

- Scaling form of the TBA eqns and the impurity free energy $\left(G(\zeta-\zeta') = \frac{1}{2\pi}\frac{1}{\cosh(\zeta-\zeta')}\right)$

$$\ln\eta_n = -2\delta_{n,1}e^{\zeta} + G[\,\ln(1+\eta_{n+1}) + \ln(1+\eta_{n-1})]$$

$$F^{(imp)} = -\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left(\zeta - \ln\frac{T_K}{T}\right)}\ln[1+\eta_1(\zeta,\frac{h}{T})]$$

T_K is only scale in the problem, thus in the scaling regime $\boxed{F^s = Tf(T/T_K, h/T)}$

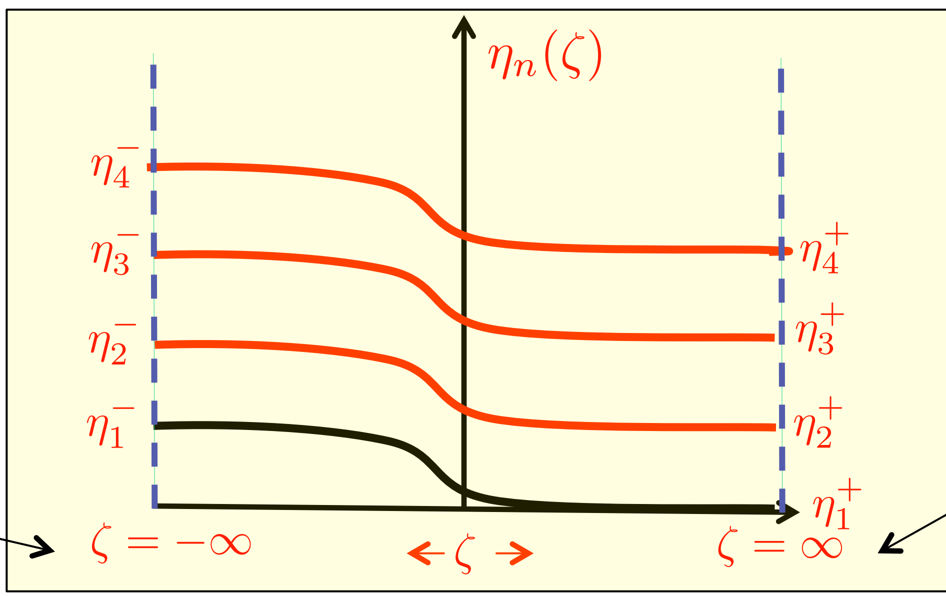
Some properties

Some properties of the solutions of the TBA eqns

1. $\eta_n(\zeta, h/T)$ is monotonically decreasing in ζ (fixed *n*).
2. $\eta_n(\zeta, h/T)$ is monotonically increasing in *n* (fixed ζ).
3. $\eta_n(\zeta, h/T)$ has finite asymptotic limits:

$$\eta_n \to \begin{cases} \eta_n^- = \dfrac{\sinh^2 (n+1)\frac{\mu h}{T}}{\sinh^2 \frac{\mu h}{T}} - 1, & \text{as } \zeta \to -\infty \\[2ex] \eta_n^+ = \dfrac{\sinh^2 n\frac{\mu h}{T}}{\sinh^2 \frac{\mu h}{T}} - 1, & \text{as } \zeta \to +\infty \end{cases}$$

High-T regime



Low –T regime

$$F^{(imp)} = -\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left(\zeta - \ln\frac{T_K}{T}\right)} \ln[1+\eta_1(\zeta, \frac{h}{T})]$$

Main contribution $\zeta \sim \ln\dfrac{T_K}{T}$

High – T regime

- Impurity behavior at high T: $T \gg T_K$

$$F^{(imp)} = -\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left(\zeta - \ln\frac{T_K}{T}\right)} \ln[1+\eta_1(\zeta, \frac{h}{T})]$$: high temperature corresponds to $\zeta \to -\infty$

Therefore:

$$F^{(imp)} \to -\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left(\zeta - \ln\frac{T_K}{T}\right)} \ln[1+\eta_1^-] = -T\ln(2\cosh\frac{\mu h}{T})$$

The free energy of an isolated spin in a magnetic field *h*

How rapidly is this point approached? Include corrections $1/\zeta, 1/\ln\zeta$

$$F^{imp} \to -T\left[\ln(2\cosh\frac{\mu h}{T}) - \frac{1}{2}\frac{\mu h}{T}\tanh\frac{\mu h}{T}\left(\frac{1}{\ln T/T_K} + \frac{1}{2}\frac{\ln\ln(T/T_k)}{\ln^2 T/T_K}\right)\right] + \cdots$$

leading to susceptibility (obtainable perturbatively)

We'll see: universal number

$$\chi^{imp} = -\frac{\partial^2 F^{imp}}{\partial h^2}\bigg|_{h=0} = \frac{\mu^2}{T}\left[1 - \left(\frac{1}{\ln T/T_K} + \frac{1}{2}\frac{\ln\ln(T/T_K)}{\ln^2 T/T_K}\right) + \frac{a}{\ln^2(T/T_K)} + O\left(\frac{\ln^2\ln T/T_k}{\ln^3 T/T_k}\right)\right]$$

In RG language: Weak Coupling fixed point behavior

Scale defined up to a constant - set $T^{wc} = e^a T_K$, then no $O(1/\ln^2)$ in $\chi^{(imp)}$

Low – T regime

- Impurity behavior at low T: $T \ll T_K$

$$F^{(imp)} = -\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left(\zeta - \ln\frac{T_K}{T}\right)} \ln[1+\eta_1(\zeta, \frac{h}{T})]$$: low temperature corresponds to $\zeta \to +\infty$

Cannot use same strategy as before -$\eta_1^+ = 0$. Need study neighborhood of point.

Expand the kernel in the integral $1/\cosh(\zeta + \ln t) = 2t \exp\zeta(1 - t^2\exp 2\zeta + t^4 \exp 4\zeta + \cdots)$, $t = \frac{T}{T_K}$

How to evaluate: $-\frac{T^2}{\pi T_K}\int d\zeta e^{\zeta} \ln\left(1+\eta_1(\zeta, \frac{h}{T})\right)$?

Recall, the total free energy

$$F = -\frac{\pi L T^2}{12} - \frac{T}{2\pi}\int d\zeta \left\{ \frac{N^e}{\cosh\left[\zeta - \ln\frac{T_K}{T} - \frac{\pi}{c}\right]} + \frac{1}{\cosh\left[\zeta - \ln\frac{T_K}{T}\right]} \right\} \ln[1+\eta_1(\zeta, \frac{h}{T})],$$

with the free energy of the electrons at temperature *T* and magnetic field *h*

$$F^{el} = -\frac{\pi L T^2}{12} - \frac{T}{2\pi}\int d\zeta \left\{ \frac{N^e}{\cosh\left[\zeta - \ln\frac{D}{T}\right]} \right\} \ln[1+\eta_1(\zeta, \frac{h}{T})]$$ in the Bethe basis

so $F^{imp} = -T f\left(\frac{T}{T_0}, \frac{h}{T}\right)$ $\quad F^{el} = -\frac{\pi L T^2}{12} - T N^e f\left(\frac{T}{D}, \frac{h}{T}\right)\Big|_{D\to\infty}$

Same behavior at ***low temp***

FL fixed point

Low – T regime

We can also evaluate electron free energy in the Fock basis

$$\frac{F^e}{L} = -\frac{T}{2\pi}\left[\int_{-(\pi D-\mu h)}^{\infty} dk\,\ln(1+e^{-k/T}) + \int_{-(\pi D-\mu h)}^{\infty} dk\,\ln(1+e^{-k/T})\right]$$

$$= -\frac{\pi T^2}{6} - \frac{(\mu h)^2}{2\pi}$$

comparing with the Bethe basis

$$\frac{F^e}{L} = -\frac{\pi T^2}{12} - \frac{T^2}{\pi}\int d\zeta e^{\zeta}\ln\left(1+\eta_1(\zeta,\frac{h}{T})\right) + O\left(\frac{T^4}{D^2}\right)$$ we have $$\int d\zeta e^{\zeta}\ln\left(1+\eta_1(\zeta,\frac{h}{T})\right) = \frac{\pi^2}{12} + \frac{(\mu h)^2}{2T^2}$$

So finally, the impurity free energy at low temperatures

$$\boxed{F^{imp} = -\frac{T^2}{\pi T_0}\int d\zeta e^{\zeta}\ln\left(1+\eta_1(\zeta,\frac{h}{T})\right) = -\frac{1}{\pi T_K}\left[\frac{\pi^2}{12}T^2 + \frac{1}{2}(\mu h)^2\right]}$$

Interpretation: strong coupling-FL fixed point

Hence: $C_v^{imp} = \frac{\pi}{6T_K}T$ $\quad \chi^{imp} = \frac{\mu^2}{\pi T_K}$ while $C_v^{el} = \frac{\pi T}{3}$ $\quad \chi^{el} = \frac{\mu^2}{\pi}$

FL specific heat *screened impurity*

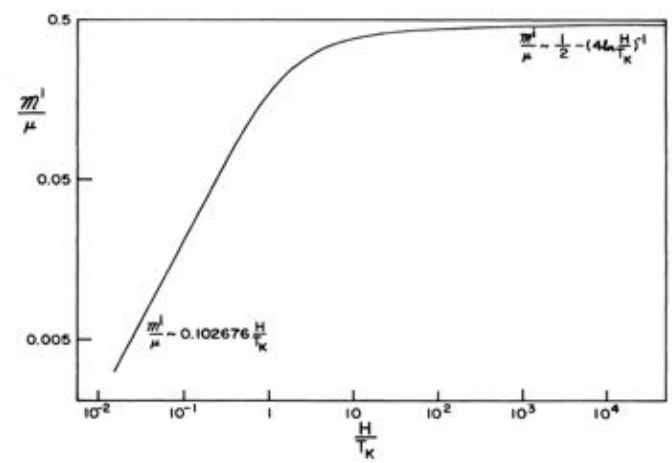
Wilson's Ratio $$R = \frac{\chi^{imp}/\chi^{el}}{C_v^{imp}/C_v^{el}} = 2$$ characterizes the strong coupling fixed point

Strong coupling scale $T^{sc} = T_K$

The cross-over

The cross-over behavior:

- The magnetization as function of the magnetic field *h* (at temperature *T=0*):



Exact analytic expression obtained using Wiener-Hopf method

$$\mathcal{M}^i \underset{h \ll T_K}{\rightarrow} \frac{\mu^2}{\pi T_K} h$$

Strong coupling regime, with scale $T^{sc} = T_K$

$$\mathcal{M}^i \underset{h \gg T_K}{\rightarrow} \mu \left[1 - \frac{1}{2 \ln \frac{\mu h}{T_h}} + \frac{\ln \ln \frac{h}{T_h}}{4 \ln^2 \frac{\mu h}{T_h}} + 0 \left(\frac{1}{\ln \frac{\mu h}{T_h}} \right)^3 + \cdots \right]$$

Weak coupling regime with scale

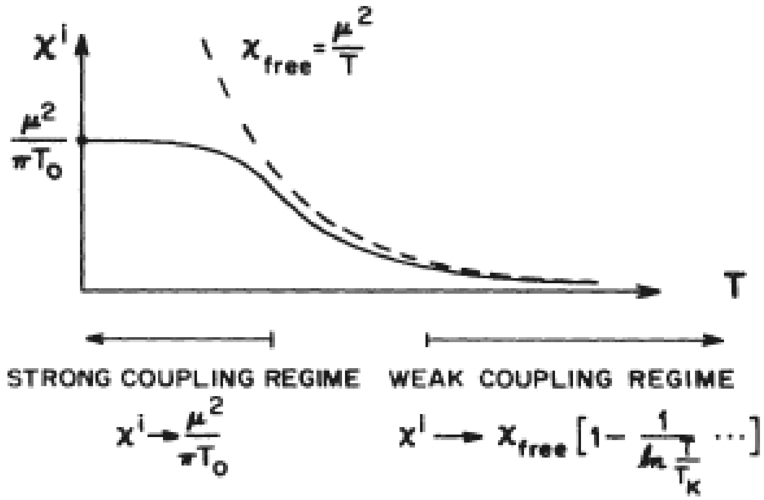
$$\boxed{T_h = \left(\frac{\pi}{e} \right)^{\frac{1}{2}} T_K}$$

defined by absorbing $1/\ln^2(h/T_h)$

This cross-over is *non-perturbative*, characterized by Wilson's magnetic universal cross-over (weak to strong) number: $W_h = \frac{T_h}{T_K} = \sqrt{\frac{\pi}{e}}$

Wilson's number

- The cross over in the temperature



Susceptibility obtained by solving TBA eqns numerically

Wish to calculate Wilson's temperature cross-over number: $W = \frac{T^{wc}}{T^{sc}}$

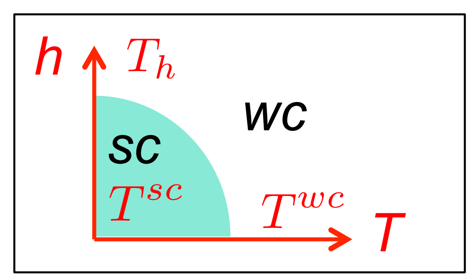
It was calculated by Wilson and requires full machinery of NRG to carry out cross-over

Difficult to obtain directly, requires solution of all η_n

Proceed indirectly: $W = \frac{T^{wc}}{T^{sc}} = \frac{T^{wc}}{T^h}\frac{T^h}{T^{sc}}$

$\frac{T_h}{T^{sc}} = \sqrt{\frac{\pi}{e}}$ connecting non-perturbatively weak to strong coupling regime

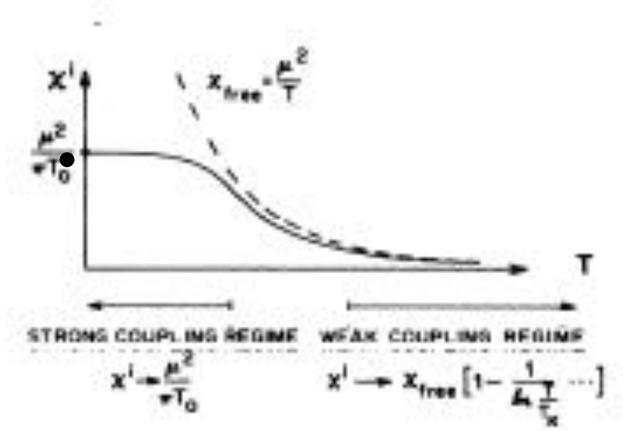
$\frac{T^{wc}}{T_h} = \frac{e^{C+3/4}}{\pi}$ computed exactly - both scales in WC regimes



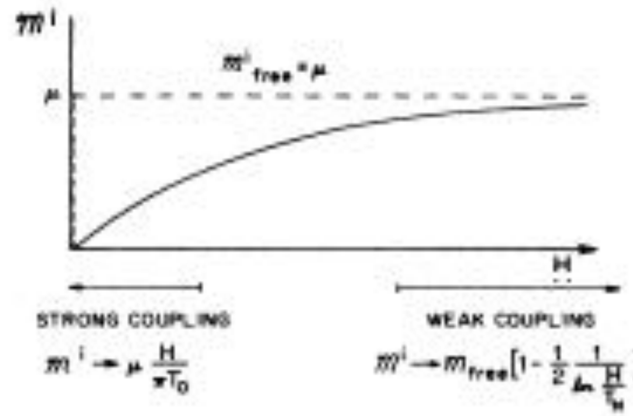
We find: $\frac{1}{4\pi}W = \frac{1}{4\pi}\frac{e^{(\mathcal{C}+1/4)}}{\sqrt{\pi}} = 0.102676...$ Wilson: $\frac{1}{4\pi}W = 0.1032 \pm 0.0005$

Thermodynamic plots

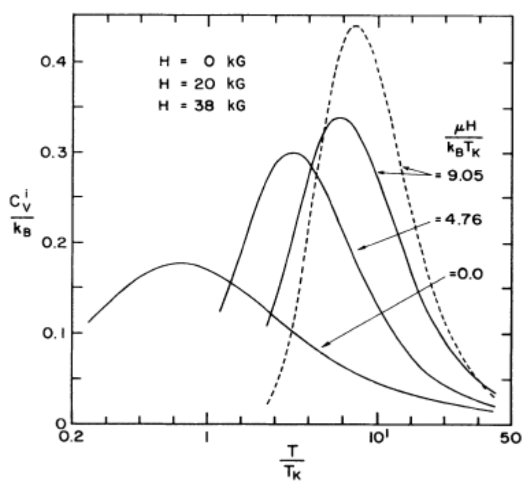
- **Impurity susceptibility**



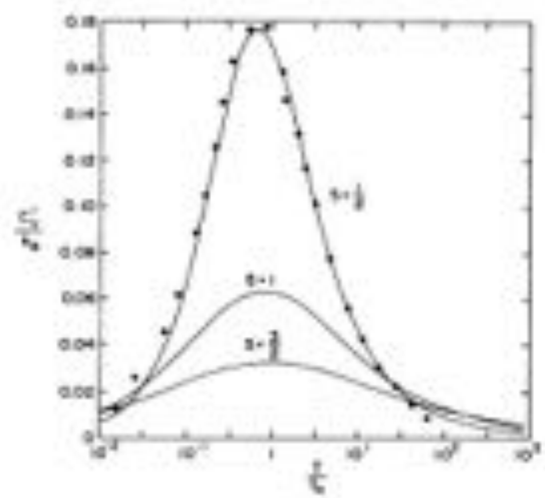
- **Impurity magnetiztion (*T*=0)**



- **The impurity specific heat**



- **The impurity specific heat**
(different spins)



High temperature or large magnetic field drives system to weak coupling (asymptotic freedom)

The Kondo Problem -summary

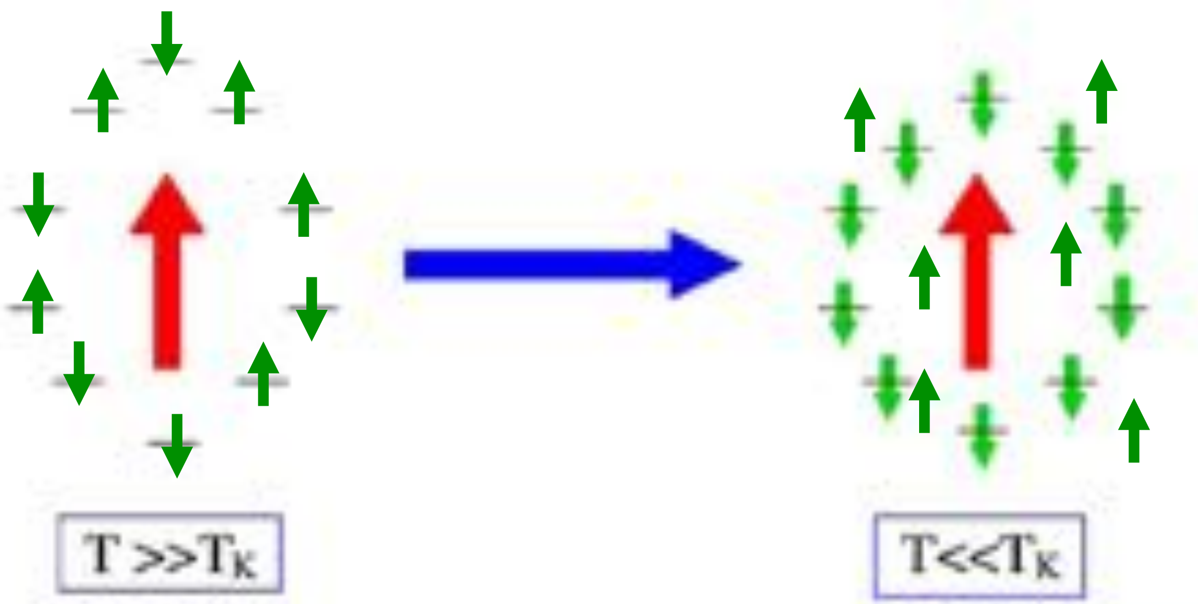
Impurity weakly coupled to electrons - unscreened

Impurity screened by excess of opposite spins (kondo cloud/resonance)

Weak coupling

$\chi \sim 1/T$

$\mathcal{S} = \ln 2$



Strong coupling - FL fixed point

$\chi \sim const$

$\mathcal{S} = 0$

- A nonperturbative energy scale emerges $\boxed{T_K \propto \exp(-1/\rho J)}$
- Below T_K impurity spin is progressively screened
- Universal scaling with T/T_K
- Conduction electrons acquire a $\pi/2$ phase shift at the Fermi level
- All initial AFM couplings flow to a single strong-coupling fixed point

Generalizations of the Kondo effect

- **The multichannel Kondo model** (Nozieres and Blandin '80)

$$H = -i\int \psi^{\dagger}_{am}(x)\partial_x\psi_{am}(x)dx + J\psi^{\dagger}_{am}(o)\vec{\sigma}_{aa'}\psi_{a'm}(o)\cdot\vec{S}$$

  - added a channel (flavor) index $m = 1\cdots f$, ($f = 1$ is the canonical case)
  - $\vec{S}$ in any spin-S representation of SU(2)

- **What is the effect of flavor?**

  - weak coupling perturbation theory is unchanged, weak coupling unstable

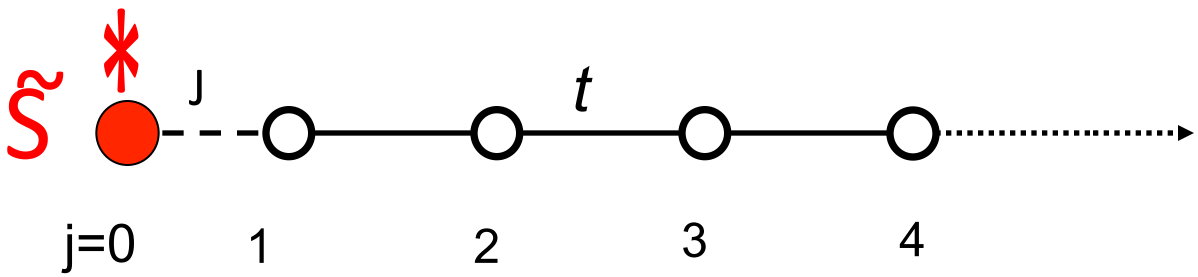
  - does it flow to strong coupling?

Generalizations of the Kondo effect

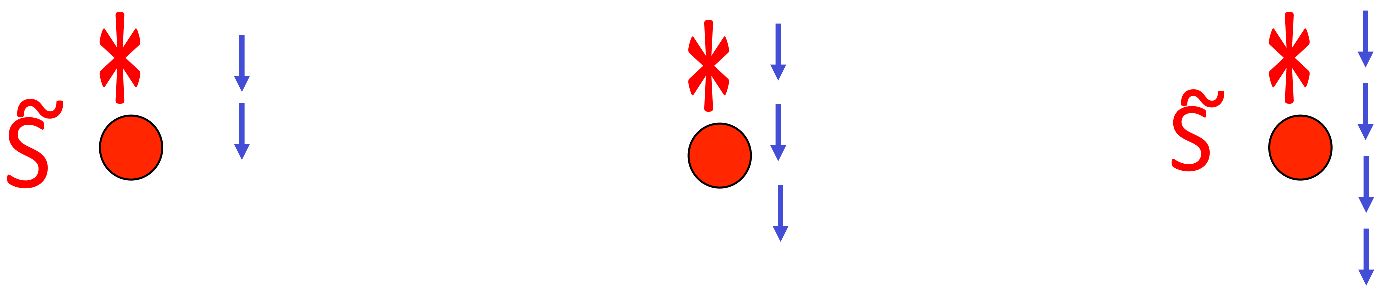
- **Stability of the strong coupling fixed point**

  - Kondo Hamiltonian on the lattice:

$$H = -t \sum_{j=0}^{\infty} [c_a^\dagger(j+1)c_a(j) + h.c] + J c_a^\dagger(0)\vec{\sigma}_{aa'}c_{a'}(0)\cdot\vec{S}$$



  - Strong coupling $J \gg t$ ground state:



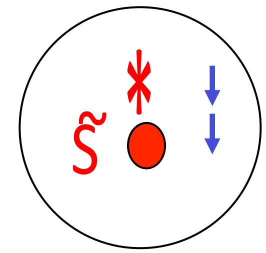
$S > \frac{1}{2}f$ underscreened

$S = \frac{1}{2}f$ screened

$S < \frac{1}{2}f$ overscreened

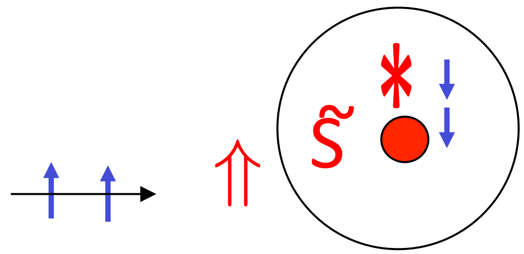
Generalizations of the Kondo effect

- **turn on the hopping perturbation - the effective spin interacts with the electrons** $\vec{S}'$



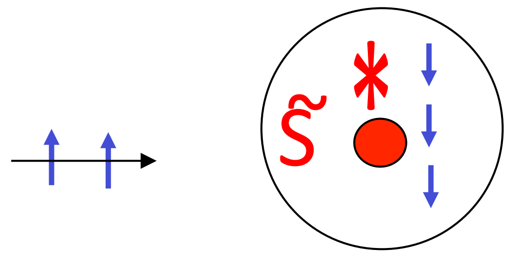
- **the effective spin interacts with up-spins only and lowers the energy** (pert. from gs)

- Underscreened :



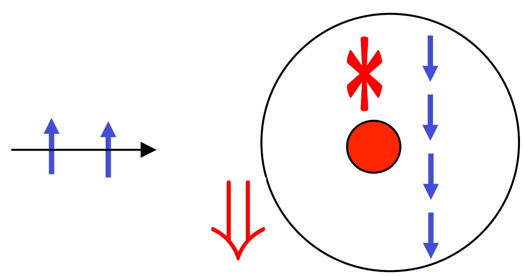
ferromagnetic interaction -
stable

- Screened



marginally ferromagnetic
stable

- Overscreened

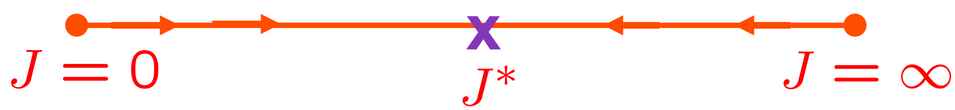


antiferromagnetic interaction
unstable

Generalizations of the Kondo effect

- **overscreened:** both strong and weak coupling fixed points unstable

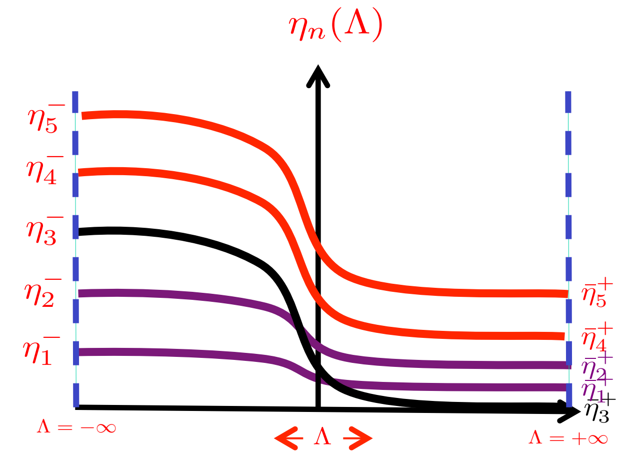
  $\longrightarrow$ *intermediate fixed point*



- **properties fixed point**: (Bethe-Ansatz '84)

$$\begin{cases} c_V \sim \left(\dfrac{T}{T_K}\right)^{\frac{4}{f+2}} \\ \chi \sim \dfrac{1}{T}\left(\dfrac{T}{T_K}\right)^{\frac{4}{f+2}} \underset{f=2}{\simeq} \ln T/T_K \\ \mathcal{S}|_{T=0} = \ln \dfrac{\sin\left((2S+1)\frac{\pi}{f+2}\right)}{\sin\frac{\pi}{f+2}} \xrightarrow[f=2]{} \ln\sqrt{2} \end{cases}$$

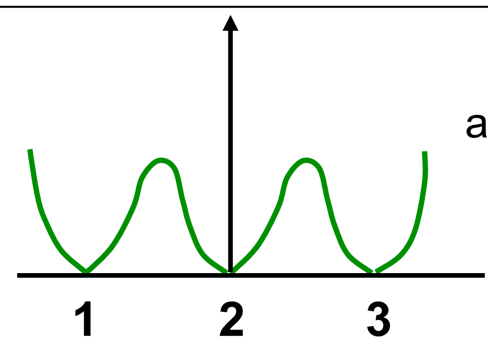
(T=0 entropy)



- **The fixed point is non-FL**

  - quasi-particles not fermion-like
  - Solitons in eff. potential

- Solitonic combination rules (origin of irrational entropy)



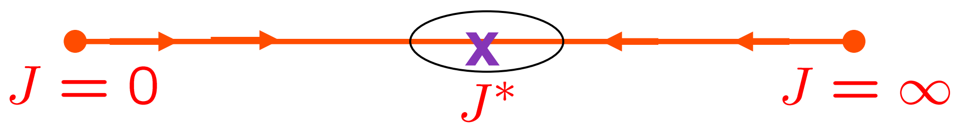
number of minima = f+1

adjacency matrix f=2 $\quad \boldsymbol{A} = \begin{pmatrix} 0, 1, 0 \\ 1, 0, 1 \\ 0, 1, 0 \end{pmatrix}$

$$\mathcal{S} = lim_{N\to\infty} \frac{1}{N} \ln Tr\ A^N$$

Generalizations of the Kondo effect

• **properties fixed point**: Boundary CFT (Affleck, Ludwig '92)



- in the neighborhood of a fixed point the theory becomes very simple

- fixed point characterized by a conformal boundary condition

- weak coupling fixed point $H_0 = -i\int \psi^\dagger_{am}(x)\partial_x\psi_{am}(x)dx$ can be expressed as a combination of spin, charge and flavor degrees of freedom $H_0 = \int[j_c j_c + J_s J_s + J_f J_f]$

$j^c(x) = \psi^\dagger_{am}(x)\psi_{am}(x)$ charge density

$J^i_s(x) = \psi^\dagger_{am}(x)(\sigma^i)_{aa'}\psi_{a'm}(x)$ spin density

$J^l_f(x) = \psi^\dagger_{am}(x)(\lambda^l)_{mm'}\psi_{am'}(x)$ flavor density

- at intermediate fixed point another rule of combination (fusion hypothesis).
To fix it need info from RG or Bathe Ansatz (or other methods that reach fixed point from microscopics)

- resistivity $R(T) \sim \sqrt{T}$ for $f=2$

• **experimental realizations of multichannel Kondo:**

Quadrupolar Kondo effect (Cox) UBe_{13} results controversial

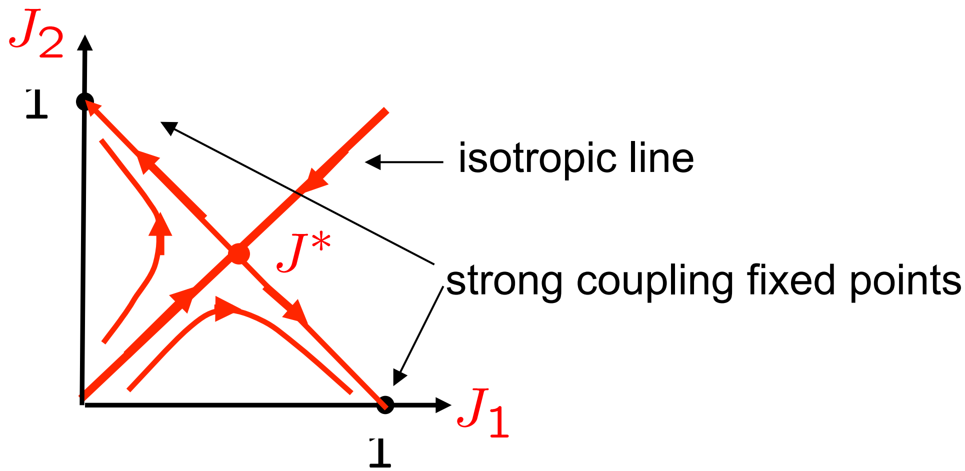
Quantum dots (Oreg, Goldgaber-Gordon)

• **system hard to realize: need exact channel symmetry**

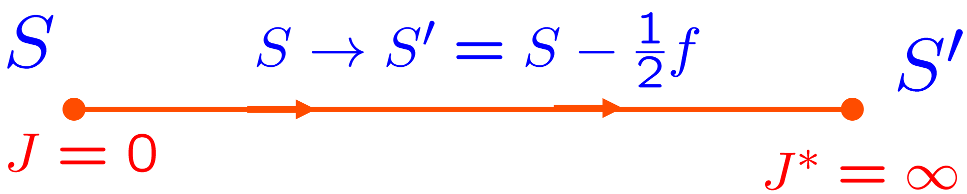
Generalizations of the Kondo effect

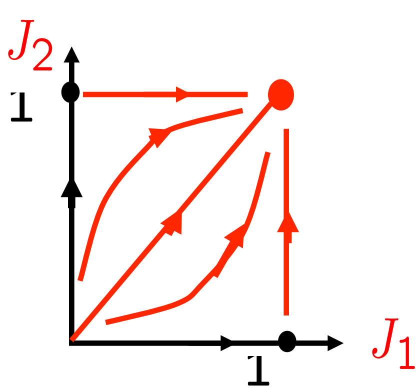
- **channel anisotropy is relevant** (around the intermediate fixed point)

$$H = -i\int \psi^\dagger_{am}(x)\partial_x\psi_{am}(x)dx + [J_1\psi^\dagger_{a1}(o)\vec{\sigma}_{aa'}\psi_{a'1}(o) + J_2\psi^\dagger_{a2}(o)\vec{\sigma}_{aa'}\psi_{a'2}(o)]\cdot\vec{S}$$



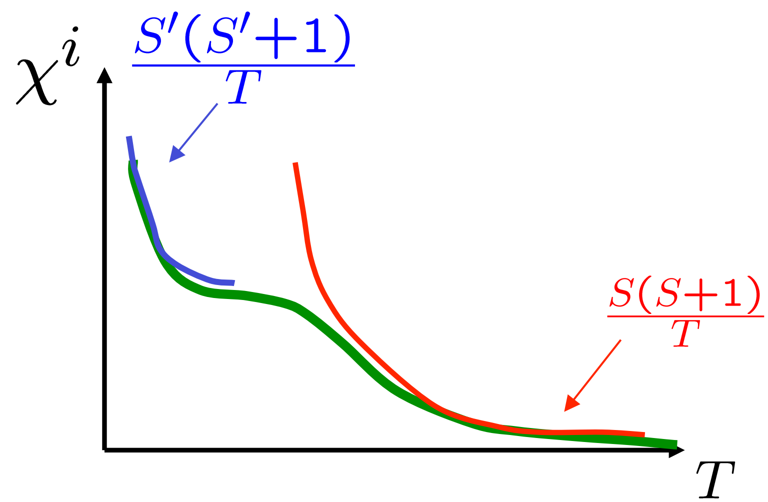
- **The underscreened fixed point**:





channel anisotropy
is irrelevant

- **The susceptibility?**

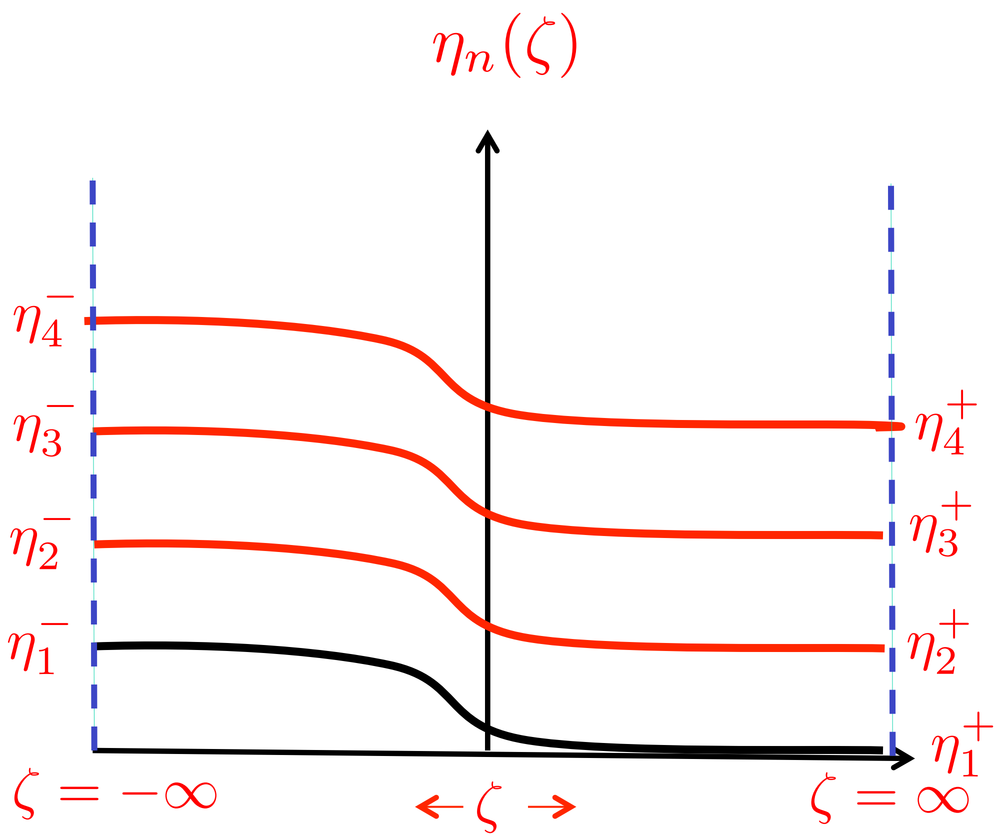


Bethe Ansatz Approach

Bethe Ansatz Approach

Bethe Ansatz Approach

The Thermodynamics



The Thermodynamics