15 Introduction to Quantum Information

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Contents

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1 Introduction

This lecture is, of course, not so closely connected with the other material of this School. Hopefully, it will give you some ideas that might come in handy when you think about the hard problems of strong correlations in electrons that you will be carefully studying in the other parts. *Quantum Information Theory* spends a lot of time thinking about the "trivial" aspects of quantum theory, the parts that you would call "kinematics" in more traditional treatments of quantum mechanics. Kinematics is the counterpart to dynamics; kinematics is the part of quantum theory where you set up your problem, get straight what basis states you will use, take care of selection rules and quantum numbers; dynamics is the part where you really solve for the behavior of your quantum system, including, as well as you can, the features of the real Hamiltonian that you want to study. I will try to convince you in this lecture that the "trivial", kinematic parts of quantum theory are really interesting in their own right, and are leading us to completely unanticipated applications of quantum theory. In fact, quantum information theory is anything but trivial, it has already generated proposed theorems that a lot of smart mathematicians worked on proving for many years (and, in one case, finally disproved). By the end of this chapter, I will at least be able to say what this particularly deep theorem was.

2 Teleportation and other quantum communication protocols

Much of quantum information theory is driven by thought experiments which explore the capabilities, in principle, for quantum systems to perform certain tasks. A few of these are very famous, like quantum cryptography, and have in fact been turned into real experiments. I will explore in detail another famous one called quantum teleportation; I will emphasize what resource question teleportation arises from, and I will show you how, by slightly changing the premise of the thought experiment, you come to another subject, which is called *remote state preparation*. Chances are you haven't heard of remote state preparation, but you will see that it is just as rich and interesting as teleportation (perhaps richer).

Quantum teleportation was first reported as a thought experiment in 1993 [1]. I have been told by two of the six authors (Bennett and Wootters) that this work arose from the question, "If Alice has a quantum state and wants to send it to Bob, how can she do it?" The presumption is that the quantum state is held in the internal state of a particle; we will often imagine that this is a *qubit* state, meaning that the Hilbert space containing the quantum state is just two dimensional. This could be because the state is that of the spin of a spin-1/2 particle, so that the two quantum basis states are "spin up" and "spin down". We will be more abstract and just give a binary labeling to these states (0/1).

The first answer to the Bennett/Wootters question is, "send the particle containing the state from Alice to Bob." So, since this was rather obvious, what Bennett and Wootters (and their co-authors Jozsa, Crépeau, Brassard, and Peres) really asked was, "what *other* method will do the same job?" In constructing their answer they were inspired by ideas in classical communication theory. Classically, there are other resources that can assist in communication tasks –

trusted third parties, authentication techniques, etc. At the basic level, many of these assistance techniques boil down to the idea that *shared classical information* between Alice and Bob, *even if it is completely random*, can greatly facilitate communication tasks. For example, it makes possible the sending of completely secret messages.

What was clear to Bennett and company was that no classical resource by itself was adequate for accomplishing the task of quantum state transmission; even an arbitrarily long classical message is inadequate for the task. The reason for this is that such classical information would have to come from somewhere, and since Alice is in possession of just one particle, the only thing she can do to make meaningful classical information is to measure this particle. But it was known since 1973 (it is called Holevo's theorem [8]) that Alice can get at most one bit of information from any measurement of a spin 1/2 particle. This is completely inadequate for Bob to reconstruct the state of the particle faithfully. A qubit quantum state in general takes two complex numbers (with a normalization constraint) to describe:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1}$$

But Bennett and company had another trick in mind, which concerned the other shared resource that is available in quantum physics: *entanglement [9].* If two separated spins have never been correlated, then their joint quantum state can only be a tensor product:

$$\begin{aligned} |\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle &= (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B) & (2)\\ &= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle) & (3) \end{aligned}$$

(I introduced a few different obvious conventions for representing this state.) This is the set of unentangled states; any pure state that cannot be written in a product form is entangled. (The situation is a little different for mixed quantum states, as I will touch on shortly.) Entanglement includes as a special case the notion of classical correlations; in fact this is represented by the mixed state

$$|\psi_{corr.}\rangle\langle\psi_{corr.}| = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|. \tag{4}$$

In words: Alice and Bob have two particles whose state is certainly the same, but with a 50% probability to be in the 0 or the 1 state. But for the general case of Eq. (2), we have known since Bell that such quantum states have correlations that cannot be mimicked by any classical theory. Thus, Bennett *et al.* were motivated to explore this shared resource for the conveyance of an arbitrary quantum state. In particular, they used the "most entangled state":

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{5}$$

This is sometimes confusingly called an "Einstein-Podolsky-Rosen" state (it is a distant relative of the state that EPR wrote down, actually due to Bohm), and less confusingly called a "Bell state".

In fact, this state proves to be very useful for the problem of conveying the state of a particle from Alice to Bob. Let us go through the details, following pp. 26-28 of [10]. We suppose that

Alice has qubit state $|\psi_0\rangle$ to "teleport" to Bob, and that they have also previously shared the Bell pair in state $|\beta_{00}\rangle$. Of course the entanglement knows nothing about the message ψ_0. But with some simple unitary transformations, this total state can be brought to a useful form for performing the communication function.

The *quantum circuit* is a basic tool of quantum information theory, so let me spend some time discussing the rules of these circuits, before going into the particulars of quantum teleportation. Figure 1 is just one example of a huge variety of quantum circuits that are used to compactly express many of the primitive operations in quantum information or quantum computation. It is somewhat analogous to a Feynman diagram of scattering theory, in that each horizontal line indicates the presence of a single qubit (so Fig. 1 represents a three-qubit operation). Time goes from left to right. The first few symbols placed on these lines indicate the actions of *quantum logic gates*. These gates perform specific unitary transformations on the quantum state. In this way, quantum information is not only about kinematics, but, in this limited way, also about dynamics. That is to say, it is understood that this unitary transformation should come from some Hamiltonian acting on the qubits at the time indicated. But quantum information theory does not care about the details of what this Hamiltonian is, or where it comes from; it only requires that the indicated state transformation can be done.

The first symbol in the circuit diagram in Fig 1, involving the first two qubits, indicates the action of the *controlled NOT* gate, or CNOT. The transformation produced by this is "classical", in the sense that it can be defined by a truth table, *viz.*,

$$\begin{aligned} 00 &\longmapsto 00 \\ 01 &\longmapsto 01 \\ 10 &\longmapsto 11 \\ 11 &\longmapsto 10\ . \end{aligned} \tag{6}$$

But it is understood that, by the linearity of the Schrödinger equation, this truth table indicates the action on any arbitrary quantum state of these two qubits, which will involve a superposition of these two basis states. The idea of the gate is that it performs a NOT (i.e., inversion of 0 and 1) on the second qubit (the "target qubit"), if the first qubit (the "control qubit") is a 1; if the control qubit is a 0, nothing is done to the target qubit.

The second gate in the quantum circuit of Fig. 1, indicated as an "H" acting on the first qubit, is called a Hadamard gate. It performs the non-classical one-qubit rotation summarized by

$$\begin{aligned} |0\rangle &\longmapsto \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ |1\rangle &\longmapsto \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\ . \end{aligned} \tag{7}$$

The next symbol encountered in Fig. 1 is not a unitary operation, but indicates that *quantum measurement* is performed on qubits 1 and 2 separately. The symbol is meant to evoke the idea of an old-fashioned electrical measuring meter; but of course the measurement of a two-level quantum system gives only the two possible (classical) outcomes "0" or "1". M_1 and M_2 are the classical variables representing these two binary outcomes; the double lines indicate that these classical values are communicated to the third qubit, controlling the action of two final one-qubit gates on this qubit. The measurement outcomes appear as an exponent in these gates; this simply means that, for example, if $M_1 = 0$, then nothing is done to the the third qubit (it has the "identity" operation performed on it), while if $M_1 = 1$, then the "X" gate is performed, which is just the NOT operation

$$\begin{aligned} 0 &\longmapsto 1 \\ 1 &\longmapsto 0\,. \end{aligned} \tag{8}$$

The action conditioned on M_2 is similar except that the "Z" gate (also known at the "π phase gate) is performed, specified by the quantum operation

$$\begin{aligned} |0\rangle &\longmapsto |0\rangle \\ |1\rangle &\longmapsto -|1\rangle\,. \end{aligned} \tag{9}$$

Both X and Z, of course, act linearly on quantum superpositions. These simple examples give essentially all the rules that are needed to interpret quantum gate diagrams.
Now, we return to the details of quantum teleportation. First, given the starting state

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\Big]\ , \tag{10}$$

local action by Alice (a "CNOT" operation between her two particles) brings this state to

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\Big]\ . \tag{11}$$

This and subsequent operations is schematized in Fig. 1. Alice then rotates her first qubit (the Hadamard rotation):

$$|\psi_2\rangle = \frac{1}{2}\Big[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\Big]\ . \tag{12}$$

Here is a very simple but revealing way of simply rewriting this state, by regrouping of qubits:

$$\begin{aligned} |\psi_2\rangle \ = \ & \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ & + \ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big]\ . \end{aligned} \tag{13}$$

Recall that the first two qubits are in Alice's possession, and the third in Bob's. In this form it is easy to see what happens if Alice now measures her two qubits. If she gets "00", then the state "collapses" (by the projection postulate) to the first term. The job is then done: the state remaining with Bob is exactly the one with which Alice started. This occurs with probability 1/4. In the other 3/4 of the cases, Alice gets a different outcome, and Bob's state is something different from $|\psi\rangle$. This is summarized so:

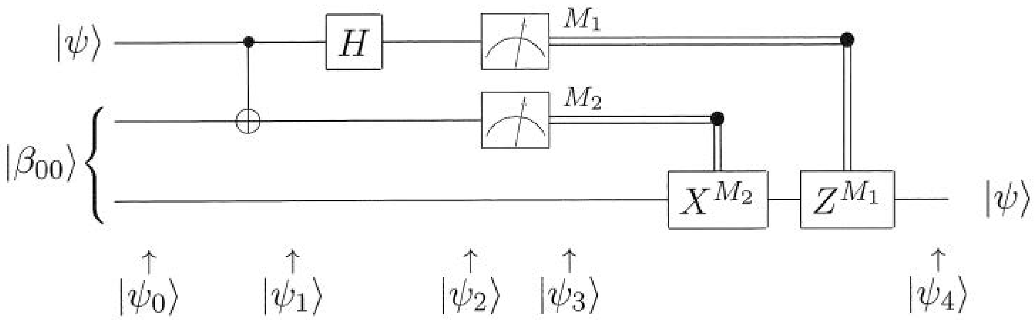

Fig. 1: *Quantum circuit summarizing the steps of quantum teleportation.*

$$00 \longmapsto |\psi_3(00)\rangle \equiv \Big[\alpha|0\rangle + \beta|1\rangle\Big] \tag{14}$$
$$01 \longmapsto |\psi_3(01)\rangle \equiv \Big[\alpha|1\rangle + \beta|0\rangle\Big] \tag{15}$$
$$10 \longmapsto |\psi_3(10)\rangle \equiv \Big[\alpha|0\rangle - \beta|1\rangle\Big] \tag{16}$$
$$11 \longmapsto |\psi_3(11)\rangle \equiv \Big[\alpha|1\rangle - \beta|0\rangle\Big] \ . \tag{17}$$

But for each of the three other outcomes, there is a simple local operation that Bob can perform which will rotate his state to $|\psi\rangle$. So, all that is required is the communication of the two measured bits from Alice to Bob, in order that he knows which rotation to perform.

To summarize the operational effect of quantum teleportation: By the prior sharing of one Bell pair, and the transmission of two classical bits, the same act of quantum communication is accomplished as the direct transmission of one arbitrary qubit state. The sharing of the Bell pair can be accomplished by the transmission of one qubit, which can be done ahead of time, as the state of this qubit is uncorrelated to the "message" qubit until the CNOT operation is performed. The classical two-bit message is also uncorrelated with the quantum message. No matter what the values of α and β, the four outcomes 00, 01, 10, and 11 occur at random with equal probability.

This is all well known and rather simple. But now, as promised, we change the premise of teleportation in an apparently trivial way. We imagine that the values of α and β are *known* to Alice. We might first wonder what "known" means; is it covered by the fact that Alice is in possession of the quantum particle with some state? The answer is "no", the particle, although it has some state, is to be unknown to Alice, supplied, say, by some third party; the particle's state may even be entangled with some other particle otherwise uninvolved in the transmission. So, in this new setting, we assume that Alice has *explicit* knowledge of α and β, by having them stored (to some high precision) in her computer memory.

This is the *remote state preparation* (RSP) setting. Alice and Bob can accomplish the task by the teleportation protocol, without making reference to the stored values α and β. But can any improvement be accomplished if Alice uses her extra knowledge?

The answer is not simple. We discovered [4] that in RSP it is necessary to discuss a resource tradeoff between entangled pairs and communication of classical bits. In the teleportation setting (Alice having no knowledge of the state), no such tradeoff is available: it was shown that, if n states are to be teleported, it is necessary to use n entangled pairs, even if it is allowed to transmit more than $2n$ classical bits; and, it is necessary to transmit $2n$ classical bits, even if it is allowed to use more than n entangled pairs.

In RSP, both tradeoffs become remarkably nontrivial. We illustrate this with one protocol that we reported, which goes in the direction of less classical communication, at the cost of higher use of entanglement. It uses another feature of two-qubit entanglement, embodied in the following equation:

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = e^{i\phi}\frac{1}{\sqrt{2}}(|\psi\psi^{\perp}\rangle - |\psi^{\perp}\psi\rangle). \tag{18}$$

Here $|\psi\rangle$ is any state of a single qubit, $|\psi^{\perp}\rangle$ is orthogonal to $|\psi\rangle$, and ϕ is a $|\psi\rangle$-dependent but irrelevant phase (which can also be absorbed into the definition of $|\psi^{\perp}\rangle$). This amounts to the interesting spin-physics statement that in the two-particle spin singlet, the two spins are opposite in any basis.

The essence of the RSP protocol that we will describe is a projective measurement by Alice. With a knowledge of α and β, Alice constructs a quantum measurement in the basis $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi^{\perp}\rangle = -\beta|0\rangle + \alpha|1\rangle$. Alice performs this measurement on her half of a singlet-entangled pair shared with Bob. With a probability of 50% she gets the outcome $\psi^{\perp}$, and in this case, because of Eq. (18), part of the job is done: At this point, Bob's state is ψ, as desired. Of course, he must be somehow informed of this fact, and something further must be done in the case of failure. Failure is not so straightforward to fix, as there is no rotation that Bob can perform to bring $\psi^{\perp}$ to ψ, given that he does not have any information about ψ. If we insist on near-certainty of success, it is not clear that any improvement can be achieved over regular teleportation.

But in fact a real improvement is possible, at least in the setting where some number of states specified by the constants $(\alpha^{(1)}, \beta^{(1)})$, $(\alpha^{(2)}, \beta^{(2)}), \ldots, (\alpha^{(n)}, \beta^{(n)})$ is given. The game will be to assume that a very large number of Alice-Bob singlets are available, and to try to successfully perform RSP on all n of these states, with the least amount of classical communication from Alice to Bob. Here is what we found [4]: Alice performs n projective measurements on her halves of the first n singlets. In all likelihood, about half of her outcomes will be $\psi^{(i)}$ and half will be $\psi^{\perp(i)}$. But if she takes many more blocks of n particles (all singlet-entangled with Bob), and repeats the same set of measurements, then eventually she will get all outcomes $\psi^{\perp(i)}$ for all n particles in that block. While this is very rare, it will happen almost certainly after she has done $2^{n+\log_2 n}$ tries. (This is the same as flipping n coins and asking if they are all "heads"; it does eventually happen.) Alice now just needs to report to Bob which of these tries was successful, and this takes $n + \log_2 n$ bits. Thus, in the limit of $n \to \infty$, the number of bits per state RSPed goes to 1. Thus, we have clearly improved on teleportation of unknown states, which cannot be done with fewer than 2 classical bits per teleportation.

This is perhaps unimpressive, given the gigantic (and diverging) amount of singlet resources that are consumed in this protocol. We found a significant improvement on this protocol, involving

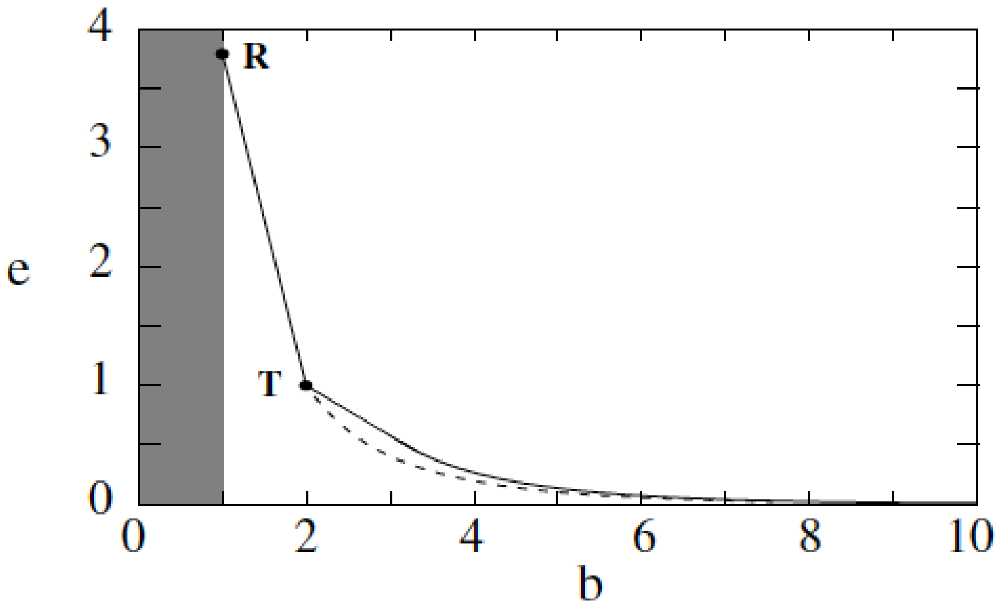

Fig. 2: *Quantum-classical tradeoff curve for Remote State Preparation, as reported in first discovery [4].*

something called entanglement recycling; I will not describe this here, but it decreases the consumption of entanglement from $2^{n+\log_2 n}$ singlets to $n(3+\frac{1}{2}\log_2 3) \approx 3.79\,n$ singlets. Fig. 2 shows that, unlike in teleportation, in RSP there is a continuous tradeoff between the number of singlets needed (called "ebits", or just "e" here) and the number of classical bits (b) that must be transmitted. Finally, within a few years the optimal tradeoff was found (Fig. 3, [5]); it was interesting that it was absolutely better than teleportation, in that it was possible to have protocols using both less entanglement (i.e., less than one singlet per RSP) and less classical communication (i.e., less than two classical bits per RSP).

It is worth noting that all of this work is "merely kinematical". It was not at all about understanding eigenstates, or about time evolution of quantum states. It all emerged from statements about the geometry of the quantum Hilbert space. But "kinematical" means anything but "trivial"; I believe that the reader of these lecture notes would find the final paper in this work, Ref. [5], a very challenging work to understand, invoking quite sophisticated statistical facts about randomly-chosen ensembles of unitary matrices.

3 More simple but deep questions about entanglement

Quantum teleportation posed more "elementary" questions about quantum entanglement that have led to extremely lengthy mathematical investigations. Bennett, Wootters, and company also wondered what would happen to teleportation if some lesser resource than perfect singlets were available. First they considered the case of states such as

$$|\psi(\alpha)\rangle = \alpha|01\rangle - \beta|10\rangle, \tag{19}$$

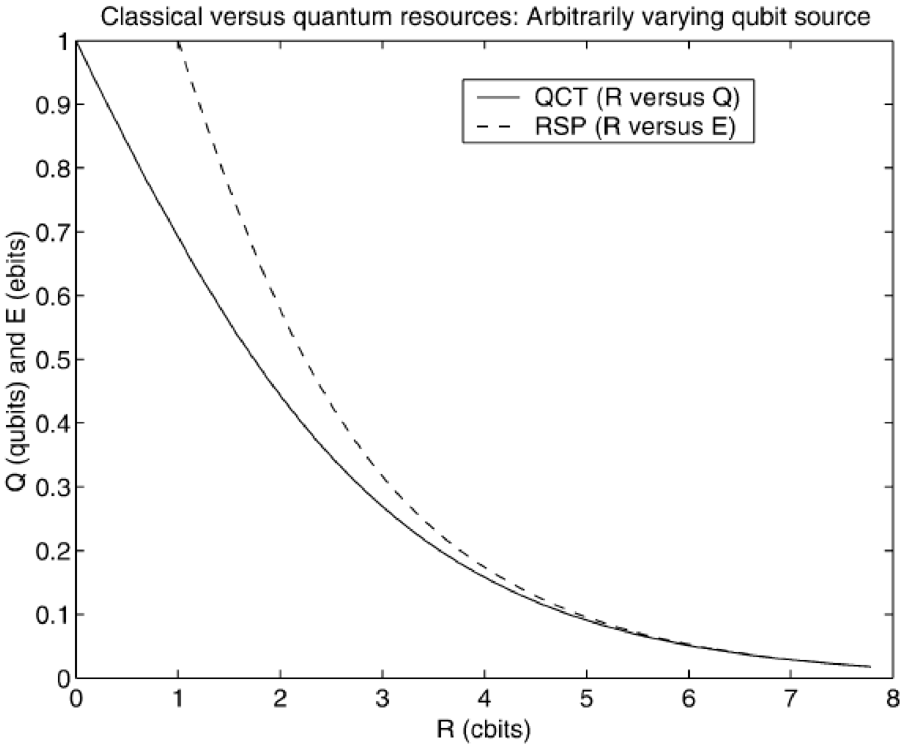

Fig. 3: *Final, optimal, quantum-classical tradeoff curve for Remote State Preparation, as calculated in [5].*

with $\alpha > 1/\sqrt{2}$, say (note the normalization condition $|\alpha|^2 + |\beta|^2 = 1$). That is, we consider states that are pure, but not maximally entangled. It was found [2] that the degree of entanglement of this state could be uniquely characterized by the entropy of the mixed state held by Alice (or Bob) alone, that is, of

$$\rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & 1-|\alpha|^2 \end{pmatrix}. \tag{20}$$

This von Neumann entropy function is just

$$H(\rho_A) = -|\alpha|^2 \log(|\alpha|^2) - (1-|\alpha|^2)\log(1-|\alpha|^2). \tag{21}$$

This function plays the following operational role: if Alice and Bob share N copies of $\psi(\alpha)$, they can, almost with certainly, *concentrate* this entanglement, obtaining $NH(\rho_A)$ copies of $\psi(1/\sqrt{2})$ (that is, of singlets). The entanglement concentration protocol is elementary, requiring only that Alice and Bob both measure the number of "0"s among their N states; no communication between them is required. It was also noted that this conversion is reversible in the limit of large N; that is, the conversion

$$N \text{ copies of } \psi(1/\sqrt{2}) \rightarrow N/H(\rho_A) \text{ copies of } \psi(\alpha) \tag{22}$$

can also be done by Alice and Bob. But the protocol (called *entanglement dilution*) is very different from that of concentration, and requires some communication (of order $\sqrt{N}$ bits) between Alice and Bob.

Returning to another operational question about teleportation, Bennett and coworkers next wondered of what use a collection of *impure* entangled states would be; by this we refer to Alice-Bob states that must be represented by density matrices – note that this is very different from the "impureness" of ρ_A above, that is just a consequence of the Alice-Bob entanglement. Said in another way, we now consider states for which the Alice and Bob parts are further entangled with a third subsystem – an "environment" – to which no one has any access. It was natural to ask whether pure entanglement could be extracted from these states, which we call ρ_{AB}. That is, we ask for a process

$$N \text{ copies of } \rho_{AB} \rightarrow D(\rho_{AB})N \text{ copies of } \psi(1/\sqrt{2}), \tag{23}$$

which may involve some classical communication between Alice and Bob, and we ask what conversion efficiency $D(\rho_{AB})$ is attainable. The process was given a new name, "distillation", because it was quickly observed that the strategies that Alice and Bob would need to employ were very different from those involved in concentration. $D(\rho_{AB})$ was referred to as the *distillable entanglement*, and it was seen to be related to various other quantities of interest in quantum communication theory. It proved, however, to be very difficult to calculate, as it remains to this day. This motivated the examination, again, of the reverse process: creating mixed states by a kind of dilution process, starting with pure entanglement. This process is defined by

$$N \text{ copies of } \psi(1/\sqrt{2}) \rightarrow N/E(\rho_{AB}) \text{ copies of } \rho_{AB}. \tag{24}$$

$E(\rho_{AB})$ is called the "entanglement cost" for forming state ρ_{AB}. It is an upper bound for the more operationally significant quantity $D(\rho_{AB})$, but there are definitely states for which they are not equal: the process of going back and forth between mixed and pure entanglement is not reversible, unlike for the pure-state case.

There is a straightforward approach to calculating $E(\rho_{AB})$, which has to do with the fact that every mixed state has an ensemble representation, that is, as mixtures of pure states:

$$\rho_{AB} = \sum_i p_i|\phi_i\rangle\langle\phi_i| \ . \tag{25}$$

Given such an ensemble $\mathcal{E}$, it is natural to define its entanglement as

$$E_{\mathcal{E}} = \sum_i p_i H(\phi_i). \tag{26}$$

This expression has a clear operational interpretation: starting with a supply of singlets, a supply of each bipartite state is created by dilution, then one of them is chosen (by the flip of a coin, so to speak). The randomness of the coin flipping creates the desired mixed state. Now, the ensemble $\mathcal{E}$ is not unique, but there is some ensemble for which the expenditure of entanglement is minimal. It is natural to conclude that

$$E(\rho_{AB}) = E_F(\rho_{AB}) = \min_{\mathcal{E}} E_{\mathcal{E}}(\rho_{AB}). \tag{27}$$

E_F, called the *entanglement of formation*, can be readily calculated for many mixed states of interest. Eq. (27) was considered so obvious in our original work that it was asserted without proof (and, in fact, with no notational distinction made between E and E_F). It became evident, however, that a proof was not obvious, and the conjecture, known as the "additivity of entanglement cost", became a central unproved proposition of quantum information theory. This conjecture received intensive scrutiny for more than 10 years. Several years ago, it was proved *false* [7]. No explicit violations of additivity have actually been exhibited, and the most explicit work on the matter shows that non-additivity will occur when the Hilbert space dimension of ρ_{AB} exceeds 3.9×10^4 (sic) [6].

4 Lessons

I hope that you have learned a few things from this lecture. First, quantum kinematics is not trivial! The structure of the quantum Hilbert space alone makes possible the posing of some very deep and complex questions, which profoundly affect how we can communicate and compute. Quantum information first identified itself as a discipline only about 20 years ago, but already there is a vast number of results, some with very profound implications for the further course of experimental physics, and some posing very deep questions for the mathematicians. We have gotten some glimpse, in our examples, of how quantum entanglement can be a unique resource for the accomplishment for a variety of concrete tasks. Theory is far, far ahead of experiment in quantum information science, but we can foresee that many of the capabilities that we theoretically envision will, someday, be a reality in the laboratory. This will give us theorists even more work to do!

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