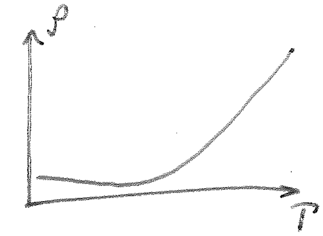
The Kondo Model and Poor Man's Scaling
(Andriy Nevidomskyy)

Historical remarks



1930s : Resistivity minimum seen in noble metals

1964 : Jun Kondo's explanation of the minimum based on the pert^{n} theory approach of the Kondo model:

$$H = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_{\substack{k,k' \\ \sigma,\sigma'}} c^\dagger_{k'\sigma'} \vec{\sigma}_{\sigma'\sigma} c_{k\sigma} \cdot \vec{S}$$

Kondo's result: $R = R_0 \left[1 - 4J\rho \ln\left(\frac{k_B T}{D}\right) + \dots \right]$

1965 : A. Abrikosov :

higher order terms $\propto \left[J\rho \ln\left(\frac{k_B T}{D}\right)\right]^n$.

Resum the series of the leading log-divergences:

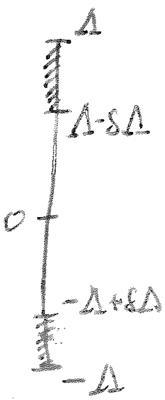
$$R = \frac{R_0}{\left[1 + 2J\rho \ln\left(\frac{k_B T}{D}\right)\right]^2}$$

Note : series is convergent as $T \to 0$ for $J < 0$ (FM), $R \to 0$.
series is divergent —"— for $J > 0$ (AFM) with instability of $\boxed{k_B T_K \sim D \exp\left(-\frac{1}{2J\rho}\right)}$ — "the Kondo temperature"

1969 – 1970 : P. W. Anderson & G. Yuval — ideas of renormalization, the "running coupling constant" $J(\Lambda)$ depends on the UV cutoff Λ.

1970 : P. W. Anderson's "Poor Man's Scaling"

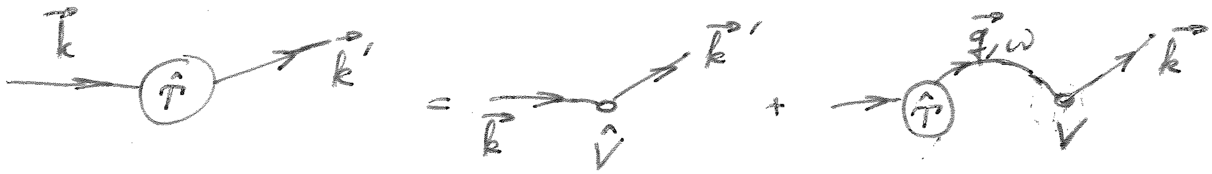
Idea of renormalization:



1. Rescale the energy cutoff $\Lambda \to \frac{\Lambda}{b} \equiv \Lambda - \delta\Lambda \qquad (b > 1)$
2. "Integrate out" the degrees of freedom in the interval $[\Lambda - \delta\Lambda, \Lambda]$, to obtain the new Hamiltonian $H \to H'(\Lambda - \delta\Lambda)$.
   Then $J \to J' = J + \delta J$
3. Rescale the energy scales back : $\omega = b\omega'$, so that $H\left(\frac{\Lambda}{b}\right) = bH'$.

Note: This last step, rescaling, omitted from P. W. Anderson's "Poor Man's Scaling" procedure

The $\hat{T}$-matrix description

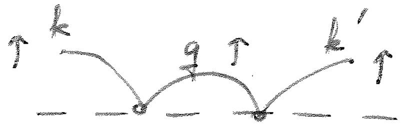


$$\hat{T}_{k'k}(\omega) = \hat{V}_{k'k} + \sum_{\vec{q}} \hat{T}_{qk} \cdot G_0(\omega, q) \cdot V_{k'q}$$

$$\hat{T}(\omega) = \hat{V} + \hat{V}\frac{1}{\omega - \hat{H}_0}\hat{T}(\omega) = \underbrace{\hat{V} + \hat{V}\frac{1}{\omega-\hat{H}_0}\hat{V}}_{\text{up to } 2^{nd} \text{ order in } V} + \hat{V}\frac{1}{\omega-\hat{H}_0}\hat{V}\frac{1}{\omega-\hat{H}_0}\hat{V} + \dots$$

Distinct physical contributions to the $\hat{T}$-matrix, depending on whether the spin-flip is allowed in the intermediate state, or not.

(1) Potential scattering, no spin-flip



+ the same with ↓ ↓ ↓

$$\hat{T}_{k'k} = \hat{V} + \sum_{q}^{\Delta > |\varepsilon_q| > \Delta - \delta\Delta} \hat{V}_{k'q} \frac{1}{\omega - \varepsilon_q + \varepsilon_k - \hat{H}_0} \hat{V}_{qk}, \quad H_0 = \sum_{k\sigma} (\varepsilon_k - \mu) c^+_{k\sigma} c_{k\sigma}$$

Consider for instance the $\vec{S}_z \cdot \vec{\sigma}_z$ part of Kondo interaction:

$$V^{\uparrow\uparrow}_{k'q} = J_z S_z c^+_{k'\uparrow} c_{q\uparrow}$$

Then $\hat{T}^{\uparrow\uparrow}_{k'k} = \hat{V}^{\uparrow\uparrow} + \delta\hat{T}^{\uparrow\uparrow}$,

$$\delta\hat{T}^{\uparrow\uparrow} = \int_{\Delta-\delta\Delta}^{\Delta} \rho(\varepsilon)d\varepsilon \frac{(J_z)^2 S_z \cdot S_z c^+_{k'\uparrow} \overbrace{c_{q\uparrow} c^+_{q\uparrow}}^{1 - c^+_q c_q = 1 - n_q \approx 1 \text{ as } T\to 0} c_{k\uparrow}}{\omega - \Delta + \varepsilon_k - \cancel{H_0}}$$

→ set to zero when all energies measured relative to $E_F = \mu$

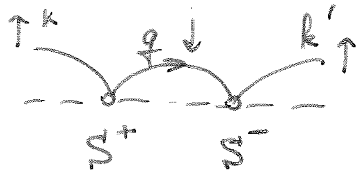
$$\simeq \frac{\rho\,\delta\Delta\,(J_z)^2 S_z^2 c^+_{k'\uparrow} c_{k\uparrow}}{\omega - \Delta + \varepsilon_k}.$$

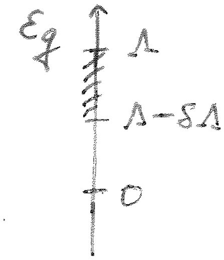
$\hat{S}_z^2 = \frac{1}{4}$ (for $s = \frac{1}{2}$).

Conclusion: w/o spin-flip, $T^{\uparrow\uparrow}$ contributes an inessential constant to the potential (spin independent) scattering.
This contribution is unimportant and can be neglected.

② Renormalization of J_z due to spin-flip processes

(a)





$$\Delta T^{(a)}_{\uparrow\uparrow} = \int_{\Lambda-\delta\Lambda}^{\Lambda} \rho(\varepsilon)\, d\varepsilon \; \frac{J_+ J_- S^- c_{k'\uparrow} \overbrace{c_{q\downarrow} \cdot c^+_{q\downarrow}}^{1-n_{q\downarrow} \approx 1 \text{ as } T\to 0} c_{k\uparrow} S^+}{\omega - \varepsilon + \varepsilon_k}$$

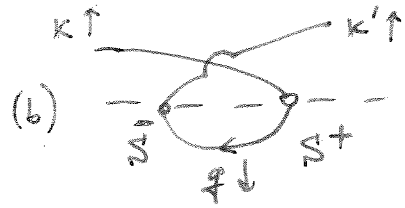
$$\cong J_+ J_- \,|\rho\,\delta\Lambda|\, S^- S^+ \, c^+_{k'\uparrow} c_{k\uparrow} \left(\frac{1}{\omega - \Lambda + \varepsilon_k} \right)$$

But $S^- S^+ = \frac{1}{2} - S_z$, therefore this term renormalizes $J_z S_z c^+_{k'\uparrow} c_{k\uparrow}$ term in the original Hamiltonian, so that

$$J_z \to J_z + \delta J_z^{(a)}, \quad \delta J_z^{(a)} = -J_+ J_- \,|\rho\,\delta\Lambda| \left(\frac{1}{\omega - \Lambda + \varepsilon_k} \right)$$

Another contribution:

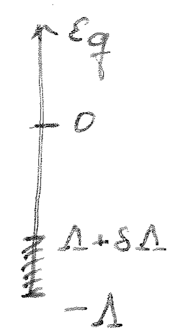
(b)



↖ hole-like intermediate state

$$\Delta T^{(b)}_{\uparrow\uparrow} = \int_{-\Lambda}^{-\Lambda+\delta\Lambda} \frac{J_+ J_- S^+ c^+_{q\downarrow} c_{k'\uparrow} \cdot c^+_{k\uparrow} c_{q\downarrow} S^-}{\omega + \varepsilon_q - \varepsilon_{k'}}$$

ε_q near lower band edge



$$\cong \frac{J_+ J_- \,|\rho\,\delta\Lambda|\, S^+ S^- \overbrace{c_{k'\uparrow} c^+_{k\uparrow}}^{\delta_{kk'} - c^+_{k\uparrow} c_{k'\uparrow}}}{\omega - \Lambda - \varepsilon_{k'}} \underbrace{\langle c^+_{q\downarrow} c_{q\downarrow} \rangle}_{f(\varepsilon_q \approx -\Lambda) \approx 1}$$

$J_z \to J_z + \delta J_z^{(b)}$, since $S^+ S^- = \frac{1}{2} + S^z$

$$\delta J_z^{(b)} = -J_+ J_- \,|\rho\,\delta\Lambda| \left(\frac{1}{\omega - \Lambda - \varepsilon_{k'}} \right)$$

Summing contributions from (a) and (b): ← ω-dependence means that interaction is retarded rather than instantaneous

$$\delta J_z = -J_+ J_- \,|\rho\,\delta\Lambda| \left[\frac{1}{\omega - \Lambda + \varepsilon_k} + \frac{1}{\omega - \Lambda - \varepsilon_{k'}} \right]$$

Since we are interested in the low-energy effective description, the frequency $\omega \ll \Delta$, and can be neglected. Similarly, we're interested in the "on shell" electrons near the Fermi surface, so $|\varepsilon_k|, |\varepsilon_{k'}| \ll \Delta$ (ε_k measured from $E_F = \mu$).

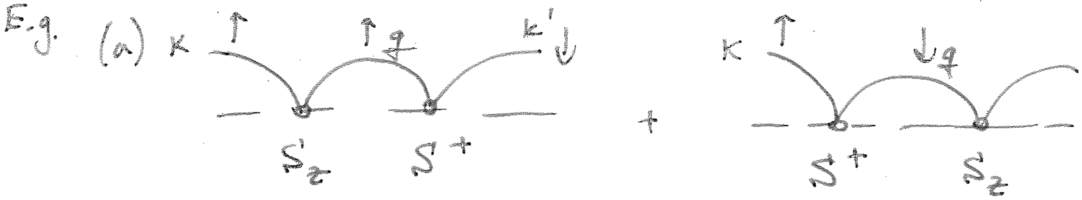
Then, $\delta J_z \simeq - J_+ J_- \rho |\delta\Delta| \left[\frac{1}{-\Delta} + \frac{1}{-\Delta} \right] = -2\rho J_+ J_- \frac{(-\delta\Delta)}{\Delta}$,

since $\delta\Delta < 0$. Rewrite as a differential eq^{n}:

$$\boxed{\frac{dJ_z}{d\ln\Delta} = -2\rho J_+ J_- \equiv -2\rho J_\pm}$$ in the isotropic case $(J_+ = J_-)$

③ Renormalization of $J_\pm$

E.g. (a)



$$\Delta T^{(a)}_{\uparrow\downarrow} = \frac{J_+ J_z |\rho\delta\Delta|}{\omega - \Delta + \varepsilon_K} \left\{ + c^+_{k'\downarrow} c_{q\uparrow} (S^+ S^z) c^+_{q\uparrow} c_{k\uparrow} - c^+_{k'\downarrow} c_{q\downarrow} (S_z S^+) c^+_{q\downarrow} c_{k\uparrow} \right\}$$

"+" because $+ J_z (c_{q\uparrow} c_{k\uparrow}) \cdot S_z$

"−" because $- J_z (c_{q\downarrow} c_{k'\downarrow}) S_z$

Since $S^+ S_z = -S^+/2$; $S_z S^+ = +S^+/2$,

both terms contribute equally to $\Delta J_+ = -J_+ J_z |\rho\delta\Delta| \frac{1}{\omega - \Delta + \varepsilon_k}$.

A similar contribution from the  diagram. In the end find

$$\delta J_\pm \simeq -J_z J_\pm \rho |\delta\Delta| \left[\frac{-2}{\Delta}\right],$$

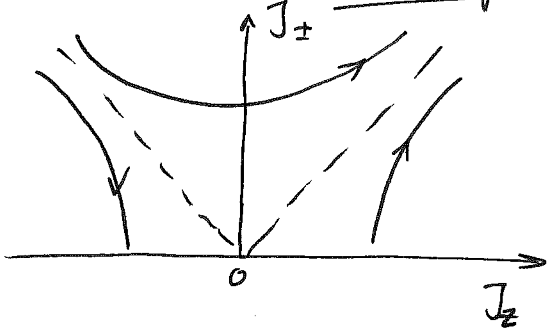
OR, $$\boxed{\frac{dJ_\pm}{d\ln\Delta} = -2\rho J_z J_\pm}$$

Renormalization Group Flow

$$g_z \equiv J_z \rho \,, \quad g_\pm \equiv J_\pm \rho$$

$$\begin{cases} \dfrac{dg_z}{d\ln \Delta} = -2g_\pm^2 \equiv \beta_z(g_\pm, g_z) \\[2ex] \dfrac{dg_\pm}{d\ln \Delta} = -2g_z g_\pm \equiv \beta_\pm(g_\pm, g_z) \end{cases}$$

Anisotropic case



Note: $\dfrac{dJ_z}{dJ_\pm} = \dfrac{J_\pm}{J_z}$,

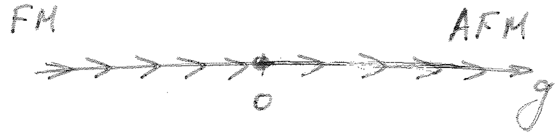
$$\Downarrow$$

$$J_z^2 - J_\pm^2 = \text{const}$$

(example of scaling law)

Isotopic case : $J_\pm = J_z$

$$\frac{dg}{d\ln \Delta} = -2g^2 < 0$$



As $\Delta \to 0$, g increases

FM coupling : $g \to 0$, so impurity decouples from the conduction sea in IR

AFM coupling : g is always relevant as $\Delta \to 0$, ∞ coupling fixed point in IR

Analogy w/ QCD : $\beta < 0$ means $g \to 0^+$ as $\Delta \to \infty$ ("asymptotic freedom")

Higher orders in perturbation theory:

$$\frac{dg}{d\ln \Delta} = -2g^2 + 2g^3 + O(g^4)$$

$$-\int_{+g}^{g^*} \frac{dg}{g^2 - g^3} = 2\ln \Delta \Big|_{D}^{\Delta^*} = -2\ln\left(\frac{D}{\Delta^*}\right)$$

$$\frac{1}{g} + \ln\left|1 - \frac{1}{g}\right| \Big|_{g}^{g^*} = -2\ln\left(\frac{D}{\Delta^*}\right) \Longrightarrow \Delta^* \sim D\sqrt{\frac{g}{1-g}}\, \exp\left(-\frac{1}{2g}\right) \approx D\sqrt{g}\, e^{-\frac{1}{2g}}$$

$$\Delta^* \sim \underbrace{k_B T_K}_{\text{Kondo scale, at which } g^* \to \infty} \sim \sqrt{J\rho}\, \exp\left(-\frac{1}{2J\rho}\right)$$

Ground State of the Kondo model

1974-75: NRG solution by Wilson

Ground state as $T \to 0$: Kondo singlet

Wilson found $\lim_{T \to 0} \dfrac{T\chi/\chi_0}{c/\gamma_0} \equiv W = 2$ (Wilson-Sommerfeld ratio)

(c.f. in conventional Fermi liquid, $W = 1$) $\quad \gamma_0 = \frac{\pi^2}{3} k_B^2 \rho$

$\chi_0 = \frac{g^2 \mu_B^2}{4} \rho$

1974: Nozières' Fermi liquid phenomenology

as $g \to \infty$ (AF case), f^a renormalizes to give additional contribution to Wilson ratio, with the result $W \to 2$ as $T \to 0$.

1980: Bethe ansatz solution by N. Andrei & (independently) by Paul Wiegmann.

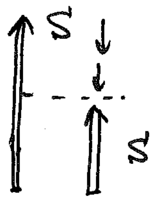
Multichannel Kondo

$$H = \sum_{k,\sigma} \sum_{\mu=1}^{K} \varepsilon_k c^+_{k\sigma\mu} c_{k\sigma\mu} + J \sum_{\mu=1}^{K} \vec{S} \cdot \vec{\sigma}_\mu , \quad \vec{\sigma}_\mu = \sum_k c^+_{k\alpha\mu} \vec{\sigma}_{\alpha\beta} c_{k\beta\mu}$$

"channels"

1980 Nozières & Blandin

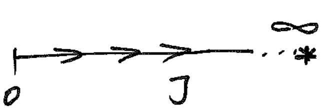
(A) Underscreened case $K < 2S$



partial screening

$S' = S - \frac{K}{2}$ spin remains at $T \to 0$ limit.

The remaining conduction el$^{\text{ns}}$ can only be "$\uparrow$" on imp-ty site, therefore they will interact ferromagnetically w/ $\vec{S}'$.

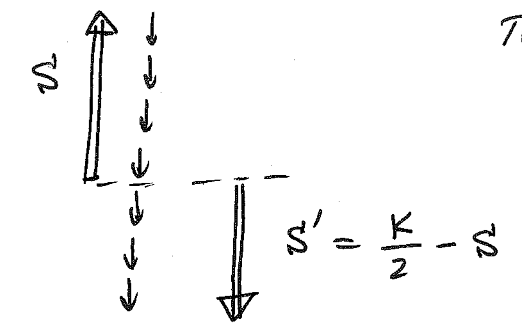


$|J'| \sim \frac{\Lambda^2}{J(\Lambda)}$ $(\Lambda \ll J)$ and J' flows to weak coupling, meaning that $J(\Lambda) \to \infty$ is stable.

(B) Perfect screening $K = 2S$

Same situation as in the single-channel Kondo. $J(\Lambda) \to \infty$, Fermi liquid ground state.

(C) Overscreened case $K > 2S$



Two-stage process as $\Delta \to 0$:

(1) Cond. electron spins "pile up" onto imp-ty site, generating effective spin $(S')_z = \frac{K}{2} - S < 0$.

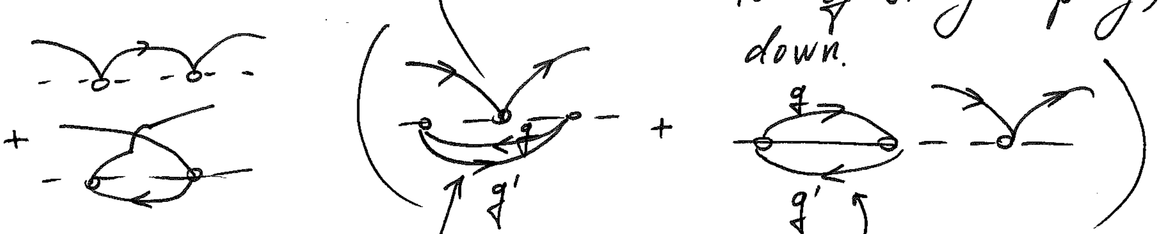
(2) This effective spin will interact with emergent strength $|J'| \sim \frac{\Delta^2}{J(\Delta)}$ in 2nd order pert. theory.

J' is AF-magnetic (unlike in underscreened case), therefore $J' \to \infty$.

But then, perturbative argument no longer holds, which means the original assumption of strong coupling, $J(\Delta) \to \infty$, breaks down.

Poor Man's Scaling:

$$\frac{d(J\rho)}{d\ln\Delta} = -2(J\rho)^2 + 2K(J\rho)^3 + \dots$$



This loop contains $\sum_{q} \sum_{\mu=1}^{K}$ "K channels"

Therefore, 3rd order term goes from $2(J\rho)^3 \to 2K(J\rho)^3$.

Attempt to find fixed point when $\beta(J^*\rho) = 0$:

$$J^*\rho = \frac{1}{K}$$

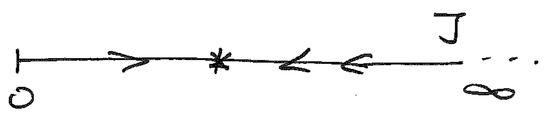
This is a meaningful expansion for large K:

$$\frac{d(J^*\rho)}{d\ln\Delta} = \underbrace{-2\frac{1}{K^2} + 2K\frac{1}{K^3}}_{\text{cancel each other}} + \#\underbrace{(J^*\rho)^2}\times\underbrace{K(J^*\rho)^2} + \dots + \#(J^*\rho)^2 \times [K(J^*\rho)^2]^2 + \dots$$

$$= 0 + \frac{\#}{K^2} \times \frac{1}{K} + \dots \frac{\#}{K^2} \times \left[\frac{1}{K}\right]^2 + \dots$$

$$= 0 + \#\frac{1}{K^3} + \frac{\#}{K^4} + \dots$$ ← convergent for $K > 1$.

Threefore, it is plausible (albeit we haven't proved it) that $\rho J^* = \frac{1}{K}$ is a stable intermediate coupling (ICFP) fixed point.



Indeed, exact Bethe ansatz sol^{n} [Andrei & Destri '84; Tsvelik & Wiegmann '84] shows that interm. coupling fixed point is stable and has

$$\chi \propto \frac{C}{T} \propto T^{\frac{4}{K+2} - 1}$$ ← unconventional non-Fermi liquid for $K > 2$.

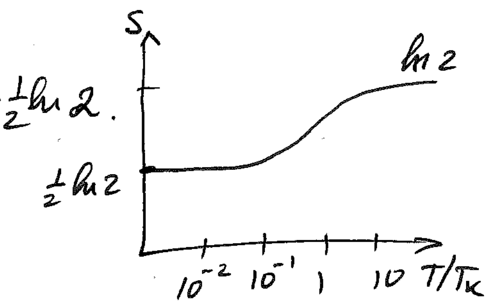
If $K=2$, $\frac{C}{T} \propto \chi \propto \ln\left(\frac{T}{T_K}\right)$. $(*)$ $\boxed{W = \frac{2(K+2)}{3}}$

Two-channel Kondo for $S=\frac{1}{2}$

1991 Ludwig & Affleck: showed that ICFP is conformal invariant, with $\frac{C}{T} \propto \chi \propto \ln\left(\frac{T}{T_K}\right)$ as from Bethe ansatz $(*)$, and

$\rho \propto \rho_0 + A\sqrt{T}$

Residual entropy $S(T\to 0) = \ln\left[2\cos\left(\frac{\pi}{K+2}\right)\right] \xrightarrow[K=2]{} \frac{1}{2}\ln 2$.

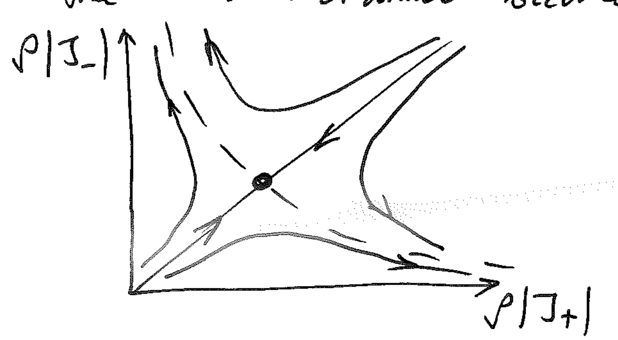


Effect of channel anisotropy

$$H_K = J_+ \vec{S}\cdot\vec{\sigma}_+ + J_- \vec{S}\cdot\vec{\sigma}_- \; , \quad J_+ \neq J_-$$

$\Delta J = J_+ - J_-$ ("channel field")

New crossover scale $T_{ch} \sim \frac{(\Delta J)^2}{T_K}$ which cuts off the RG flow: the stronger channel tends to strong-coupling fixed point, and the weaker channel becomes irrelevant:



Effect of Hund's coupling

Perfectly screened K-channel Kondo for K spins $\frac{1}{2}$:

$$H=\sum \varepsilon_k c^{+}_{k\sigma\mu} c_{k\sigma\mu} - J_H\left(\sum_{\mu=1}^{K} \vec{s}_\mu\right)^2 + J\sum_{\mu=1}^{K} \vec{s}_\mu\cdot\vec{\sigma}_\mu$$

Limit $J_H=0$: K copies of single-impurity Kondo, $T_K\sim D e^{-\frac{1}{2J\rho}}$

Limit $J_H=\infty$: "Big spin" $S=\frac{K}{2}$ screened by K channels.

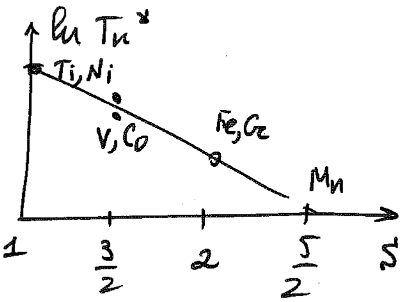
Turns out J renormalized $\rightarrow J^*=\frac{J}{K}\equiv\frac{J}{2S}$

$$\Downarrow \quad T_K^*\propto e^{-\frac{2S}{2J\rho}} \simeq T_K\cdot\left(\frac{T_K}{\Delta_0}\right)^{2S-1}$$

where $\Delta_0=J_H S$ is typical Hund's scale

Or, $T_K^*\sim \Delta_0\left(\frac{T_K}{\Delta_0}\right)^{2S}$

$$\ln T_K^* \sim \ln\Delta_0 - (2S)\ln\left(\frac{\Delta_0}{T_K}\right)$$



Note: Lowest order diagrams in Poor Man's scaling are the same as in ordinary Kondo with K channels:

$$\frac{d(J^*\rho)}{d\ln\Delta} = -2(J^*\rho)^2 + 2K(J^*\rho)^3$$

Note 2: $J^*=\frac{J}{2S}$ is obtained as follows.

By Wigner-Eckart theorem, $J\langle S S_z|\vec{s}_\mu|S S_z\rangle = J\,cst\,\langle S_z|\vec S|S_z\rangle$

Sum both parts over $\sum_{\mu=1}^{K}$ and recall that $\sum_\mu \vec s_\mu=\vec S$:

$$J\langle S_z|\vec S|S_z\rangle = cst\cdot KJ\langle S_z|\vec S|S_z\rangle$$

$$\Downarrow \quad \boxed{cst=\frac{1}{K}}$$

So $J\sum_{\mu=1}^{K} \vec s_\mu \vec\sigma_\mu = J\cdot cst\ \vec S\cdot\left(\sum_\mu\vec\sigma_\mu\right) \equiv J^*\vec S\cdot\left(\sum_\mu\vec\sigma_\mu\right)$

Therefore $J^*=cst\cdot J=\frac{J}{K}\equiv\frac{J}{2S}$ Q.E.D.

Note 3: Hund's coupling itself becomes a running coupling constant, but its renormaliz^n is weak: $\frac{d(J_H\rho)}{d\ln\Delta}=4(J\rho)^2 J_H\rho$, for $J_H S<\Delta<D$.

Kondo Resonance Narrowing

T_K^* exponentially suppressed with S $\Rightarrow$ Abrikosov–Suhl resonance much narrower as a result of Hund's coupling

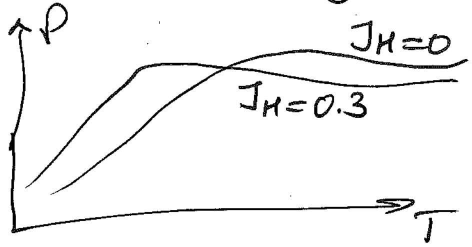
Exp-tal evidence:

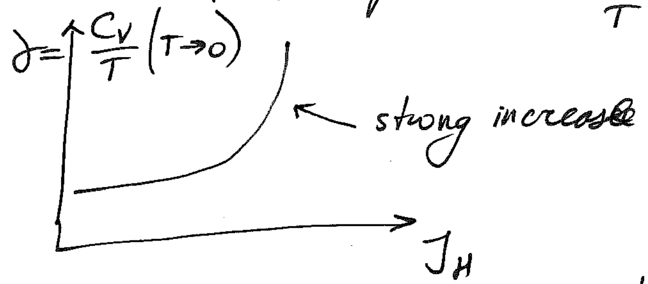
(1) Numerical RG on 2-orbital Anderson model, Pruschke & Bulla (2005)

For $J_H = \frac{U}{10}$, Kondo resonance is narrower by a factor 10^{-7}.

(2) Iron-based SC (DFT + DMFT)

Hund's coupling drastically effects the "coherence scale" in transport & in $\frac{C_V}{T}\Big|_{T\to 0}$





See Haule & Kotliar, New J. Phys (2009)