Finite – temperature Lanczos method

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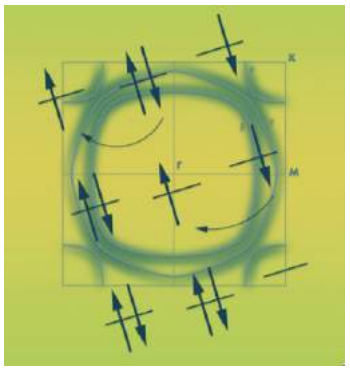

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Outline

Introduction to ED –based method:

- models, lattices

Lanczos based methods:

- T = 0 ground state and correlation functions
- T > 0 Lanczos method: static and dynamical properties
- T >> 0 microcanonical Lanczos method
- advantages and limitations

Examples of application:

- t – J model: low T - thermodynamics, spectral functions
- frustrated Heisenbeeg model: triangular lattice ➔ spin liquid
- many-body localization: high-T dynamics

Numerical calculations - simulations

The only computer experiments worth doing are those
that yield a surprise ! (V. Arnold ~ 1970)

Why to investigate model (strongly correlated) systems numerically:

- to discover new phenomena
- to get new ideas for construction of the theory
- to confirm existing theories, concepts, experiments
- to establish the right model for experiments
- to produce reliable model results
- to disprove wrong theories, speculations
- to do something
- to spend computer time

Many – body quantum lattice models of strongly correlated electrons

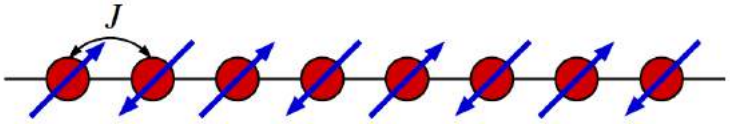
Limitation of methods based on ED (exact diagonalization):

The size of the Hilbert space: K quantum degrees/ site

$$\Longrightarrow N_{st} = K^N$$

N lattice sites

Heisenberg model: spin systems



$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \qquad S = 1/2 \qquad K = 2 \qquad N_{st} = 2^N$$

taking into account **conserved quantities and symmetries:**

total spin: $S^z_{tot} = 0 \Longrightarrow N_{st} = \binom{N}{N/2} < 2^N$

translation + rotation symmetry of the lattice : $\tilde{N}_{st} = N_{st}/N \cdots$

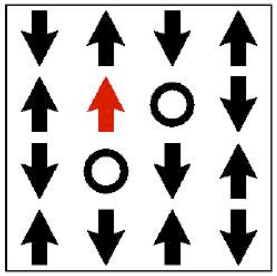
practical limit today: $N_{st} < 10^9 = 1GB \Longrightarrow N = 36$

spins + holes

t - J model: doped insulator, HTC cuprates..

$$H = -\sum_{\langle ij\rangle s}(t_{ij}\tilde{c}^{\dagger}_{js}\tilde{c}_{is} + \text{H.c.}) + J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j$$

$$\tilde{c}^{\dagger}_{is} = (1 - n_{i,-s})c^{\dagger}_{is}$$



$$K = 3 \Longrightarrow N_{st} \propto 3^N$$

projected fermion operators

symmetries: $S^z_{tot} + \mathbf{q} + N_h$ at present: $N = 32 \quad N_e = 4$

Hubbard model: Mott insulators, correlated metals..

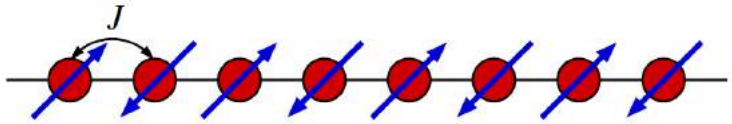
$$H = -t\sum_{\langle ij\rangle s}(c^{\dagger}_{js}c_{is} + c^{\dagger}_{is}c_{js}) + U\sum_i n_{i\uparrow}n_{i\downarrow}$$

electrons with spin

$K = 4 \Longrightarrow N_{st} \propto 4^N$ at present: $N = 20$

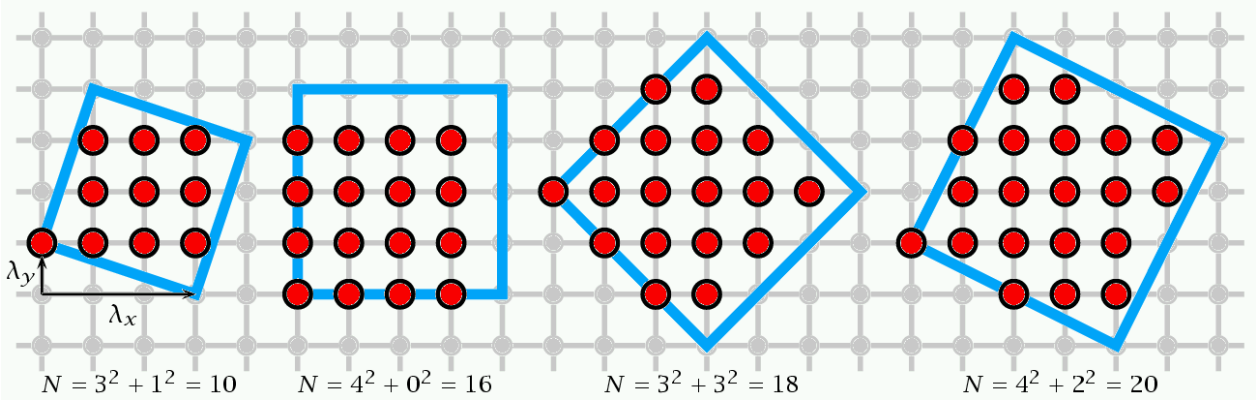
Lattices

1D systems:



there are more powerful methods than for statics (DMRG ..), but not necessarily for **T > 0 dynamical quantities**

2D systems: periodic boundary conditions
square lattice, tilted (+ distorted) square lattice,
frustrated lattices (triangular, Kagome..)



Exact diagonalization

A) Full ED: T > 0 statics and dynamics

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle \qquad \text{Memory} \propto N_{st}^2 \qquad \text{Operations} \propto N_{st}^3$$

$$\langle A\rangle = Z^{-1}\sum_{n=1}^{N_{st}} \mathrm{e}^{-E_n/T}\langle\Psi_n|A|\Psi_n\rangle \qquad \Longrightarrow N_{st} < 20000$$

Example - Heisenberg S=1/2 model: $N \leq 20$

B) Lanczos – based ED: T = 0 ground state, g.s. dynamics

sparse Hamiltonian matrix ! Basis states only locally connected

$$\text{Memory} \propto \text{Operations} \propto N_{st}$$

Lanczos ED technique

Diagonalization of sparse matrices: - power method

C. Lanczos (1950), R. Haydock, V.Heine (1970)

1) start with (random) normalized vector in Hilbert space $|\phi_0\rangle$

2) generate orthogonal Lanczos vectors $L_M = \{|\phi_m, m = 1, M\}$

Krylov space $\{|\phi_0\rangle, H|\phi_0\rangle, \cdots, H^M|\phi_0\rangle\}$

$$H|\phi_0\rangle = a_0|\phi_0\rangle + b_1|\phi_1\rangle$$

$$H|\phi_i\rangle = b_i|\phi_{i-1}\rangle + a_i|\phi_i\rangle + b_{i+1}|\phi_{i+1}\rangle \qquad 1 \leq i \leq M$$

3) diagonalization of (real) tridiagonal matrix *M x M*: real a_i, b_i

$$|\psi_j\rangle = \sum_{i=0}^{M} v_{ji}|\phi_i\rangle \qquad \boxed{\langle\psi_i|H|\psi_j\rangle = \epsilon_j\delta_{ij}} \quad \textbf{exact !} \quad i, j \leq M$$

approx. eigenfunctions

convergence for ground state: $M < 60 \ll N_{st}$

T = 0 dynamics via Lanczos

dynamical (auto)correlation function for operator A

$$C(\omega) = \int_0^\infty \mathrm{e}^{i\omega^+ t} \langle \Psi_0 | A^\dagger(t) A | \Psi_0 \rangle = \langle \Psi_0 | A^\dagger \frac{1}{\omega^+ + E_0 - H} A | \Psi_0 \rangle$$

1) start the Lanczos procedure with normalized $|\tilde{\phi}_0\rangle = \frac{1}{\alpha} A|\Psi_0\rangle$

2) generate Lanczos functions $|\tilde{\phi}_j\rangle \Longrightarrow |\tilde{\psi}_j\rangle, \; j = 1, M$ and $\tilde{a}_j, \tilde{b}_j$

3) $$C(\omega) = \cfrac{\alpha^2}{\omega^+ + E_0 - \tilde{a}_0 - \cfrac{\tilde{b}_1^2}{\omega^+ + E_0 - \tilde{a}_1 - \cfrac{\tilde{b}_2^2}{\omega^+ + E_0 - \tilde{a}_2 - \ldots}}}$$

frequency moments: $\mu_l = -\frac{1}{\pi} \int_{-\infty}^{\infty} \omega^l \mathrm{Im} C(\omega) d\omega = \alpha^2 \langle \tilde{\phi}_0 | (H - E_0)^l | \tilde{\phi}_0 \rangle$

exact ! for given $|\Psi_0\rangle$ provided $l \leq \tilde{M}$

Matrix elements via Lanczos

How to evaluate matrix element for any operators A, B ?

$$W_{kl} = \langle n|H^k B H^l A|n\rangle$$

Perform 2 x Lanczos $M > (l,k)$!

$$L_M : |\phi_0\rangle \cdots |\psi_j\rangle, \epsilon_j \qquad \tilde{L}_M : |\tilde{\phi}_0\rangle \propto A|\phi_0\rangle \cdots |\tilde{\psi}_j\rangle, \tilde{\epsilon}_j$$

$$W_{kl} = \sum_{i=0}^{M}\sum_{j=0}^{M} \langle\phi_0|\psi_i\rangle\langle\psi_i|B|\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|A|\phi_0\rangle(\epsilon_i)^k(\tilde{\epsilon}_j)^l$$

exact ! $M > (l,k)$

Proof: projectors $P_M = \sum_{i=0}^{M} |\psi_i\rangle\langle\psi_i|$ $\tilde{P}_M = \sum_{i=0}^{M} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$

$$W_{kl} = \langle\phi_0|P_M H P_M H \ldots H P_M B \tilde{P}_M H \ldots \tilde{P}_M H \tilde{P}_M A|\phi_0\rangle$$

FTLM - T > 0 static quantities

Operator A : J. Jaklič, PP (1994)

$$\langle A\rangle = \sum_{n=1}^{N_{st}} \langle n|e^{-\beta H}A|n\rangle \Big/ \sum_{n=1}^{N_{st}} \langle n|e^{-\beta H}|n\rangle$$

1) Calculate *M* Lanczos steps for each $|n\rangle$, and corresponding $|\psi_i^n\rangle$

$$\langle A\rangle = Z^{-1}\sum_{n=1}^{N_{st}}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\langle n|H^k A|n\rangle \qquad Z = \sum_{n=1}^{N_{st}}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\langle n|H^k|n\rangle$$

$$\langle n|H^k A|n\rangle = \sum_{i=0}^{M}\langle n|\psi_i^n\rangle\langle\psi_i^n|A|n\rangle(\epsilon_i^n)^k$$ correct $k < M$, extended to $k > M$!

$$\langle A\rangle = Z^{-1}\sum_{n=1}^{N_{st}}\sum_{i=0}^{M} e^{-\beta\epsilon_i^n}\langle n|\psi_i^n\rangle\langle\psi_i^n|A|n\rangle \qquad Z = \sum_{n=1}^{N_{st}}\sum_{i=0}^{M} e^{-\beta\epsilon_i^n}\langle n|\psi_i^n\rangle\langle\psi_i^n|n\rangle$$

reproduces high-T expansion up to the order M **!**

reproduces correctly also T = 0 result !

$$M \sim 50 : \; |\psi_0^n\rangle \sim |\Psi_0\rangle$$

$$\langle A\rangle = \sum_{n=1}^{N_{st}} \langle n|\Psi_0\rangle\langle\Psi_0|A|n\rangle \Big/ \sum_{n=1}^{N_{st}} \langle n|\Psi_0\rangle\langle\Psi_0|n\rangle = \langle\Psi_0|A|\Psi_0\rangle / \langle\Psi_0|\Psi_0\rangle$$

2) Replace full sum with random sampling

$$N_{st} \text{ states } |n\rangle \Longrightarrow R \text{ random states } |r\rangle, \quad R \ll N_{st}$$

$$\tilde{A}_r = \langle r|e^{-\beta H} A|r\rangle / \langle r|e^{-\beta H}|r\rangle$$

single random state

$$|r\rangle = \sum_{n=1}^{N_{st}} \eta_{rn}|n\rangle$$

$$= \sum_{n,m=1}^{N_{st}} \eta_{rn}^* \eta_{rm} \langle n|e^{-\beta H} A|m\rangle \Big/ \sum_{n,m=1}^{N_{st}} \eta_{rn}^* \eta_{rm} \langle n|e^{-\beta H}|m\rangle$$

random value, sign

$$\bar{A}_r = \sum_{n=1}^{N_{st}} |\eta_{rn}|^2 \langle n|e^{-\beta H} A|n\rangle \Big/ \sum_{n=1}^{N_{st}} |\eta_{rn}|^2 \langle n|e^{-\beta H}|n\rangle$$

$$|\eta_{rn}|^2 = 1/N_{st} + \delta_{rn}$$

$$\langle n|e^{-\beta H}|n\rangle = Z_n \text{ and } \langle n|e^{-\beta H}A|n\rangle = Z_n A_n$$ uncorrelated

$$\bar{A}_r = \langle A\rangle(1 + \mathcal{O}(1/\sqrt{\bar{Z}}))$$

$$\bar{Z} = e^{\beta E_0}\sum_n Z_n = \sum_{n=1}^{N_{st}}\langle n|e^{-\beta(H-E_0)}|n\rangle$$

statistical error: effective number of states Z(T)

$$T \to \infty : \bar{Z} \longrightarrow N_{st}$$

At high $T >> 0$ single random state is enough to evaluate $\langle A\rangle$!

Random sampling: $1 \leq R \ll N_{st}$

$$\langle A \rangle = \frac{N_{st}}{ZR} \sum_{r=1}^{R} \sum_{j=0}^{M} e^{-\beta \epsilon_j^r} \langle r|\psi_j^r\rangle \langle \psi_j^r|A|r\rangle \qquad Z = \frac{N_{st}}{R} \sum_{r=1}^{R} \sum_{j=0}^{M} e^{-\beta \epsilon_j^r} |\langle r|\psi_j^r\rangle|^2$$

$$\boxed{\delta\langle A\rangle/\langle A\rangle = \mathcal{O}(1/\sqrt{R\bar{Z}})}$$

3) Conserved quantities (energy, specific heat..): $[H, A] = 0$

$$\boxed{\langle A \rangle = \frac{N_{st}}{ZR} \sum_{r=1}^{R} \sum_{j=0}^{M} e^{-\beta \epsilon_j^r} |\langle r|\psi_j^r\rangle|^2 A_j^r}$$

just few additional lines
in the Lanczos code !

$\langle r|\psi_j^r\rangle = v_{j0}^r$

no need to calculate (and store) Lanczos wf.,
overlap given by diagonalization of 3-diagonal matrix !

FTLM - T > 0 dynamical (auto)correlations

$$\chi''(\omega) = \pi(1 - e^{-\beta\omega})C(\omega) \qquad C(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^{+\infty} dt e^{i\omega t} C(t)$$

$$C(t) = \langle A^\dagger(t)A(0)\rangle = \frac{1}{Z}\sum_n \langle n|e^{(-\beta+it)H} A^\dagger e^{-iHt} A|n\rangle$$

$$C(t) = Z^{-1}\sum_{n=1}^{N_{st}}\sum_{k,l=0}^{\infty} \frac{(-\beta+it)^k}{k!}\frac{(-it)^l}{l!}\langle n|H^k A^\dagger H^l A|n\rangle$$

perform 2 x Lanczos with $|\phi_0\rangle = |n\rangle, |\phi_0\rangle \propto A|n\rangle$, exact $l, k \leq M$

extrapolation: $j, k > M$

$$C(t) = Z^{-1}\sum_{n=1}^{N_{st}}\sum_{i,j=0}^{M} e^{-\beta\epsilon_i^n} e^{it(\epsilon_i^n - \tilde{\epsilon}_j^n)} \langle n|\psi_i^n\rangle\langle\psi_i^n|A^\dagger|\tilde{\psi}_j^n\rangle\langle\tilde{\psi}_j^n|A|n\rangle$$

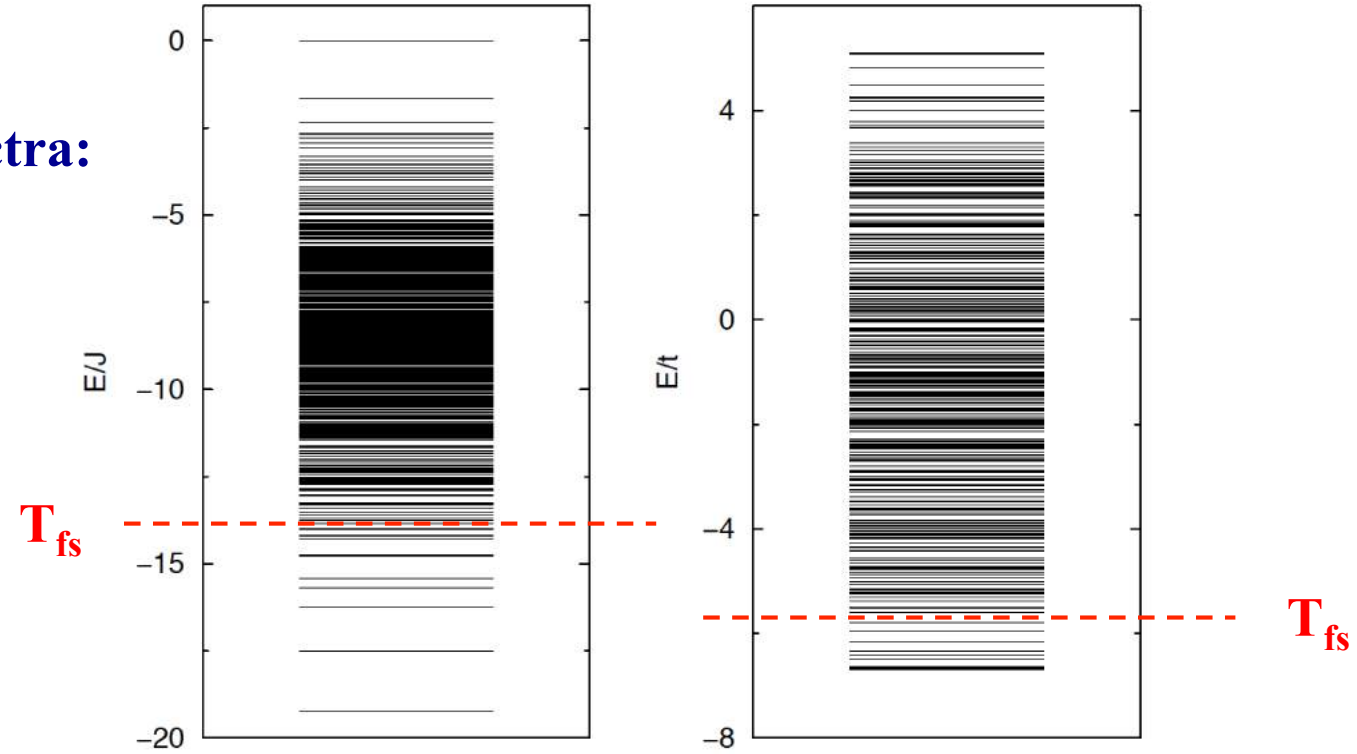
$$C(\omega) = \frac{N_{st}}{ZR}\sum_{r=1}^{R}\sum_{i,j=1}^{M} e^{-\beta\epsilon_i}\langle r|\psi_i^r\rangle\langle\psi_i^r|A^\dagger|\tilde{\psi}_j^r\rangle\langle\tilde{\psi}_j^r|r\rangle\delta(\omega - \tilde{\epsilon}_j^r + \epsilon_i^r)$$

Finite – size effects: T_{fs}

Which size or T is large enough to represent the macroscopic result ?

Effective number of MB states: $\bar{Z} = \mathrm{Tr}\, e^{-\beta(H-E_0)} = Z^*(T_{fs}) > 1$

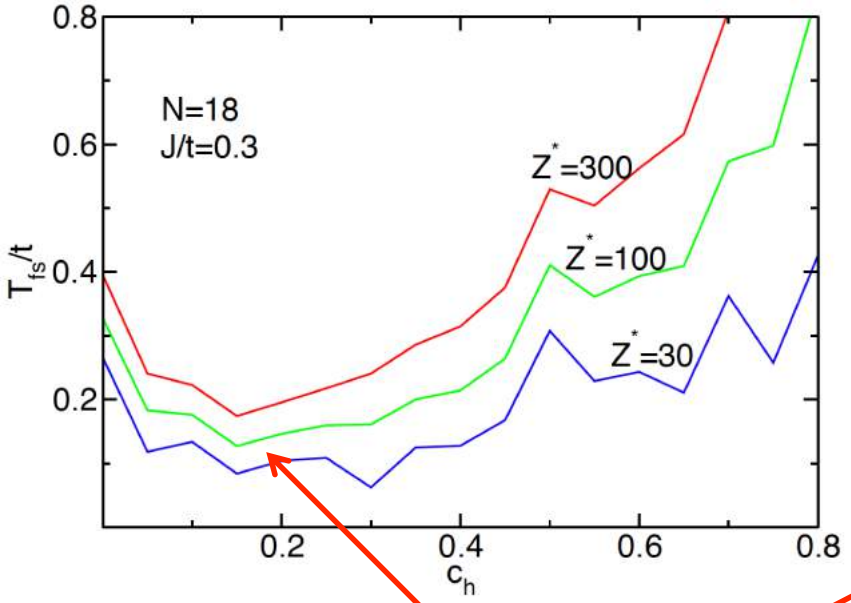
MB spectra:



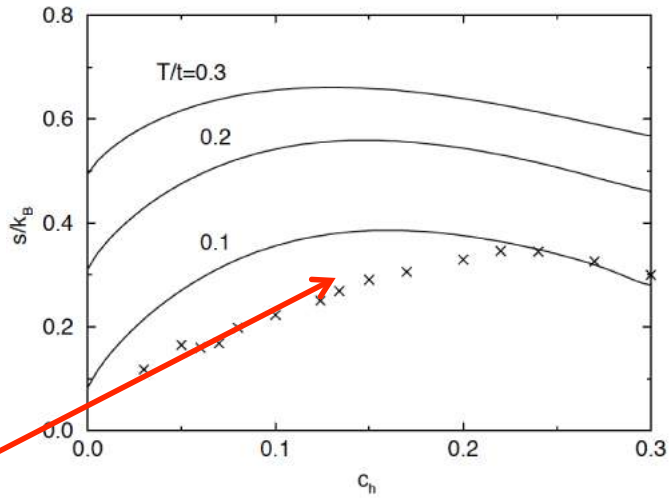
2D Heisenberg: *N=16*

2D t-J model: *N=10* N_h*=2*

2D $t-J$ model: doped model more frustrated, **lower T_{fs}** !



entropy



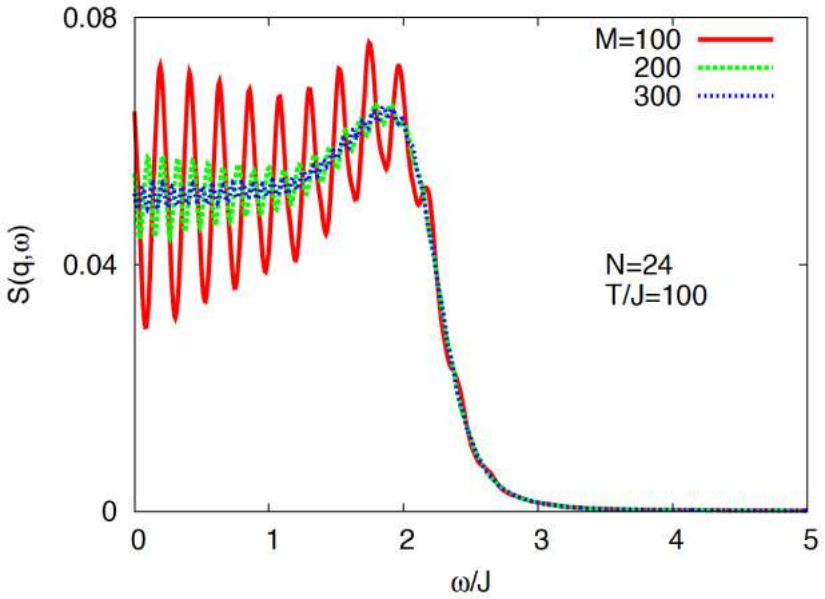
'optimal' doping (for SC T_c...)

more frustration, better for FTLM and ED-based methods:

- shorter correlation length
- large density of low lying MB states, therefore lower T_{fs}
- frustrated systems: macroscopic-like results for $T > T_{fs}$

How many Lanczos steps?

1D Heisenberg model $S(q = \pi,\omega)$: $T >> 0$



$T >> 0$ oscillations in spectra:

- Lanczos generates equidistant 'levels' in the middle MB spectrum
- no information content ?

MCLM: Microcanonical Lanczos method

Dynamical correlations at high $T >> 0$: might be nontrivial ?

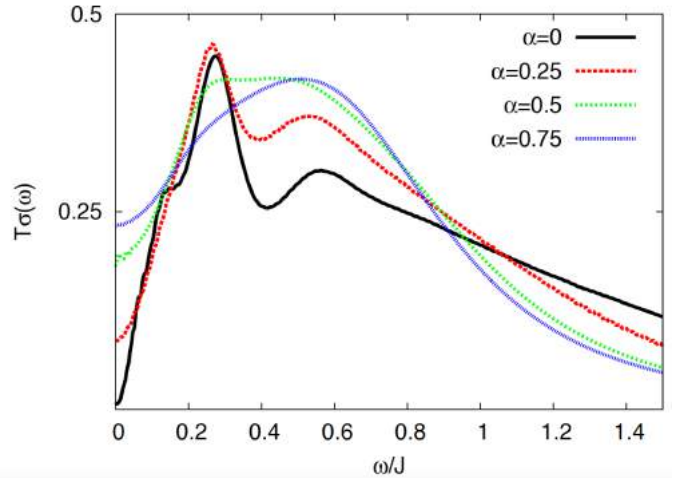
if **dynamics singular:** high – ω required, i.e., large $M > 100$

instead FTLM use MCLM: no wf. needed !

$$C(\omega,\lambda) = \langle\Psi_\lambda|A^\dagger \frac{1}{\omega^+ + \lambda - H} A|\Psi_\lambda\rangle \qquad \lambda = \langle H\rangle(T)$$

wf. with ~ good energy: $\sigma_E/\Delta E < 10^{-3}$ $\sigma_E^2 = \langle\Psi_\lambda|V|\Psi_\lambda\rangle$

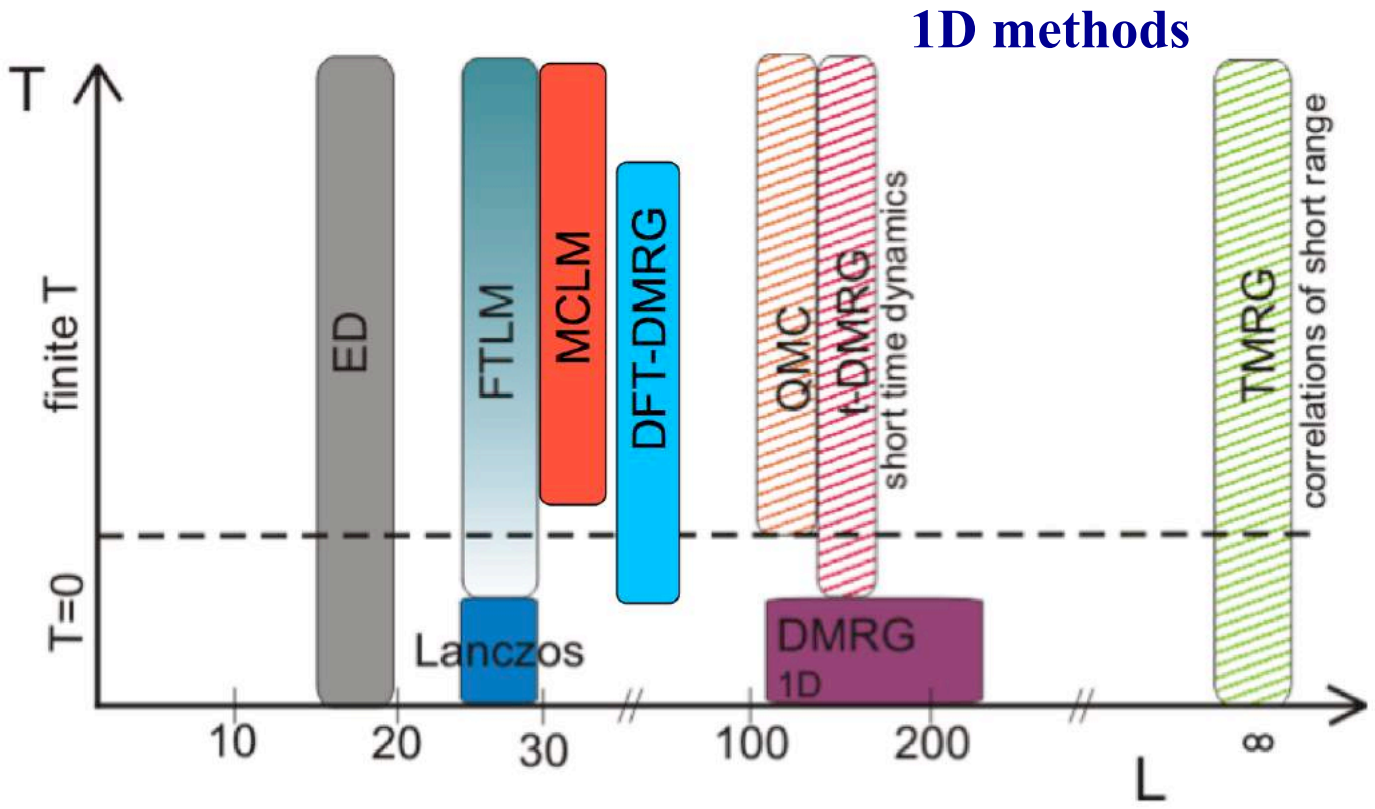
perform Lanczos with $V = (H-\lambda)^2$ $M_1 \sim 1000$



Heisenberg: frustrated J_1-J_2 chain

dynamical (spin) conductivity

Numerical methods for T > 0 dynamics



Lanczos – based T > 0 methods

FTLM: T > 0

- represent (optimal ?) interpolation between T=0 Lanczos and HTE
- most valuable for **dynamical correlations**, where not many alternatives
- easy to implement: analogous to T = 0 method + matrix elements
- easy to control: very pedagogical ! code without evident errors usually OK !!
- best for frustrated systems with high entropy !
- similar methods: low-T LM, FTD-DMRG
- limitations: storage of Lanczos wf., full Hilbert space N_{st}

MCLM: T >> 0

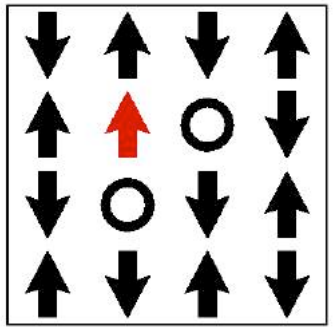
- requirements the same as T =0 (g.s.) code, possibly M >> 100
- interesting for dynamics of nontrivial systems (integrable, disordered MB..)
- similar methods: typicality approach..

t – J model

interplay : electron hopping + spin exchange

single band model for **strongly correlated electrons**

$$H = -\sum_{i,j,s} t_{ij} \tilde{c}^{\dagger}_{js} \tilde{c}_{is} + J \sum_{\langle ij \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j)$$



$t_{ij} = t$ n.n. hopping

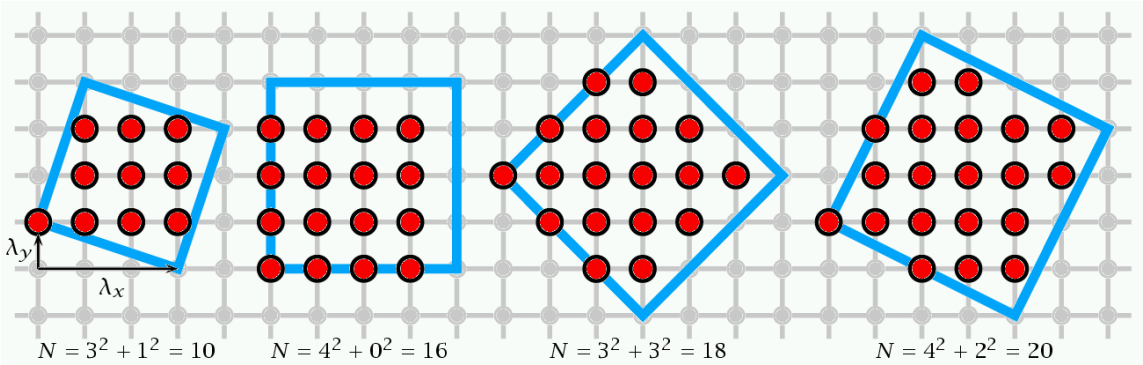
$t_{ij} = t'$ n.n.n. hopping

etc.

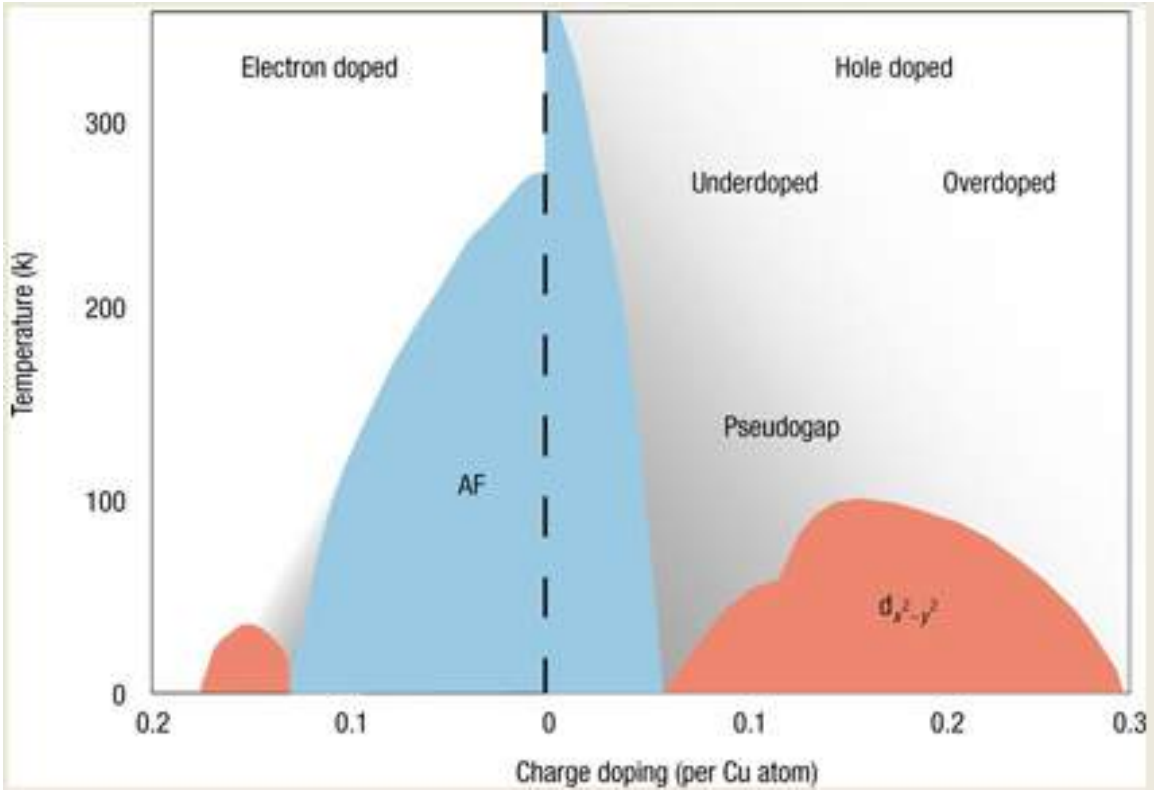
$$\tilde{c}^{\dagger}_{is} = (1 - n_{i,-s}) c^{\dagger}_{is}$$

projected fermionic operators:

no double occupation of sites

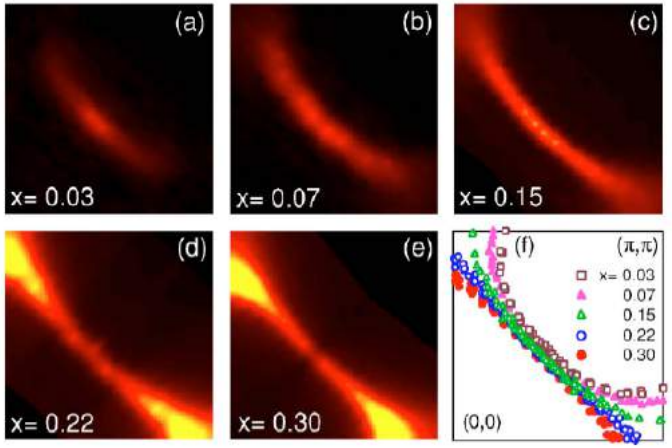


Cuprates: phase diagram

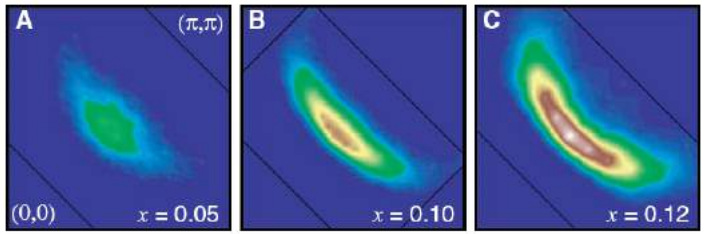


Hole-doped cuprates: ARPES

Fermi surface reconstruction:

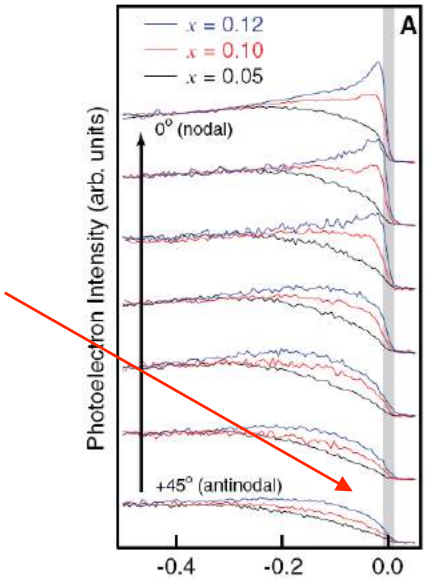


$La_{2-x}Sr_xCuO_4$: Yoshida et al 06



Na-CCOC : K.Shen et al 05

Pseudogap:



Spectral functions

$$G(\mathbf{k},\omega) = -i\int_0^\infty dt e^{i(\omega+\mu)t}\langle\{\tilde{c}_{\mathbf{k}s}(t),\tilde{c}^\dagger_{\mathbf{k}s}\}_+\rangle$$

projected operators

$$G(\mathbf{k},\omega) = \frac{\alpha}{\omega-\zeta_{\mathbf{k}}-\Sigma(\mathbf{k},\omega)} \qquad |\Sigma(\mathbf{k},\omega\to\pm\infty)| \propto 1/\omega$$

$\alpha = (1+c_h)/2$ normalization

$\zeta_{\mathbf{k}} = \int d\omega\omega A(\mathbf{k},\omega)/\alpha$

'free' term

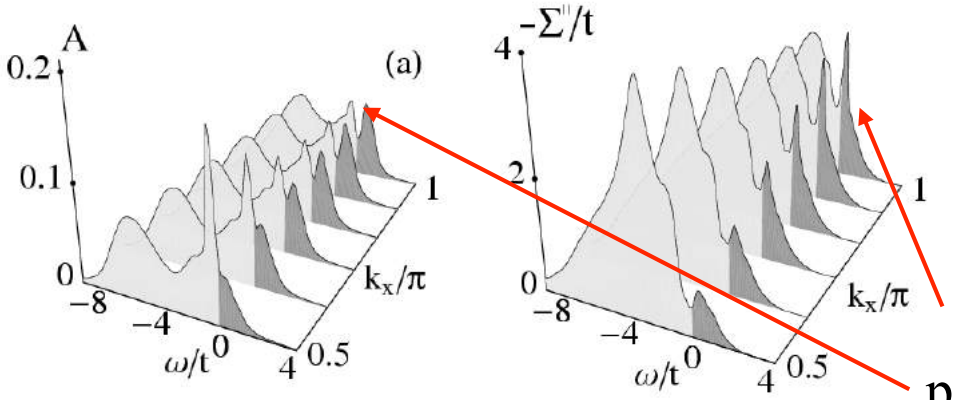
Finite size lattice:

Continuous **k:** $t_{ij} \to \tilde{t}_{ij} = t_{ij}\exp(i\vec{\theta}\cdot\vec{r}_{ij})$ $\mathbf{k} = \mathbf{k}_l + \vec{\theta}$

Regularization: with FTLM calculate $G(\mathbf{k},\omega) \longrightarrow \Sigma(\mathbf{k},\omega)$

$\longrightarrow$ average $\Sigma(\mathbf{k},\omega)$ over $\delta k \sim 0.3$ $\longrightarrow$ $\boxed{G(\mathbf{k},\omega)}$

Pseudogap: spectral function and self energy along the 'Fermi line'



t - t' - J model:

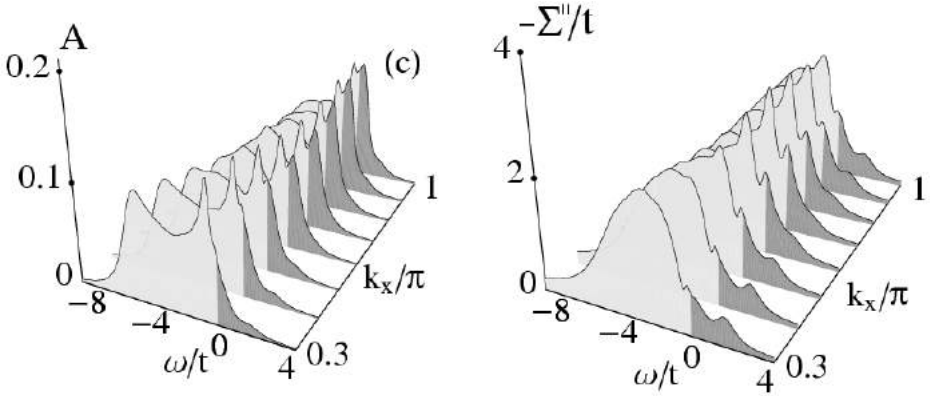
low doping: $c_h = 0.05$

pseudogap contribution

calculated for lowest $T = T_{fs}$!
macroscopic result ?

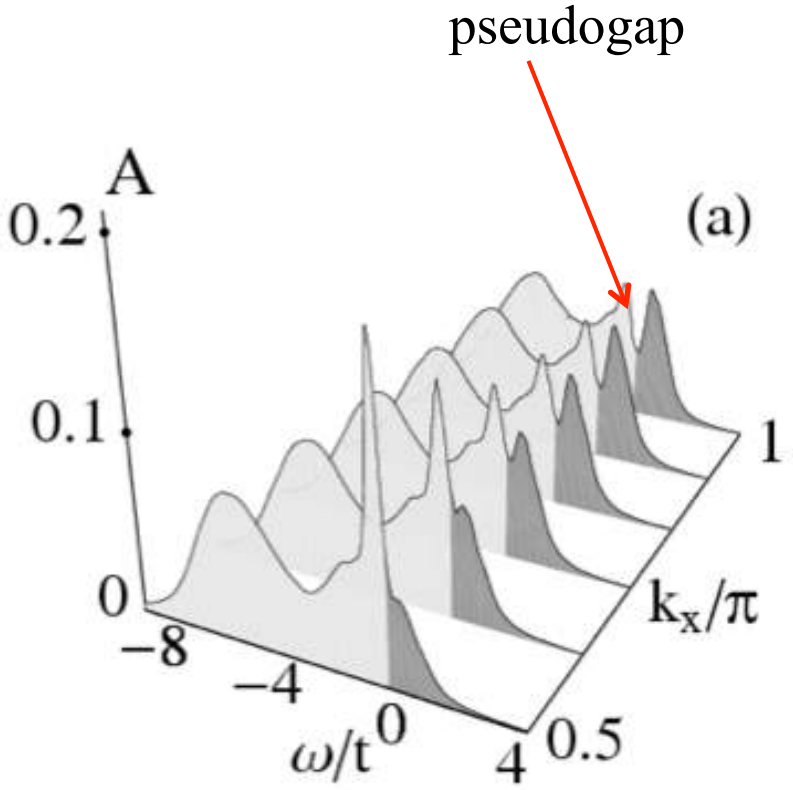
$$-\Sigma''_{MFL}(\mathbf{k}, \omega \sim 0) \sim a_{\mathbf{k}} + b_{\mathbf{k}}|\omega|$$

marginal FL damping

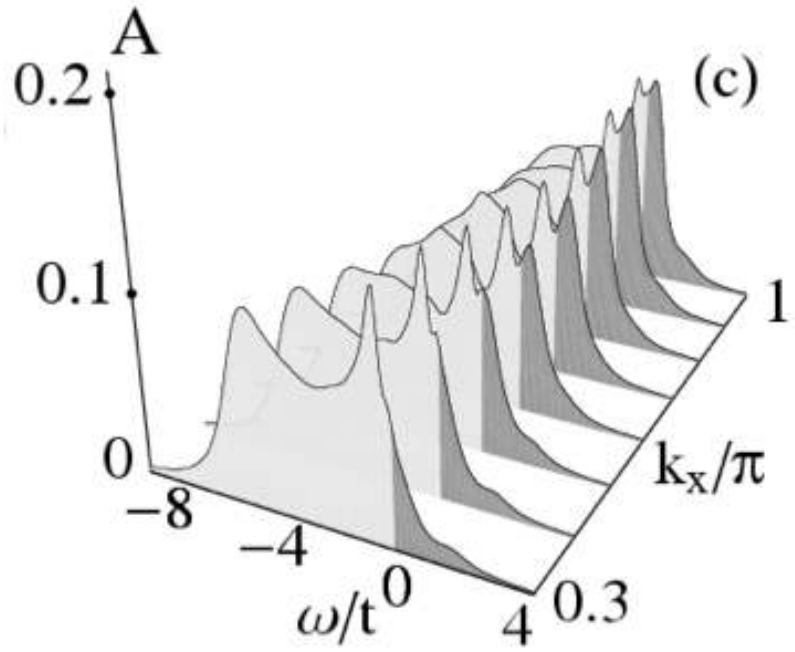


intermediate (optimum) doping:

$c_h = 0.17$

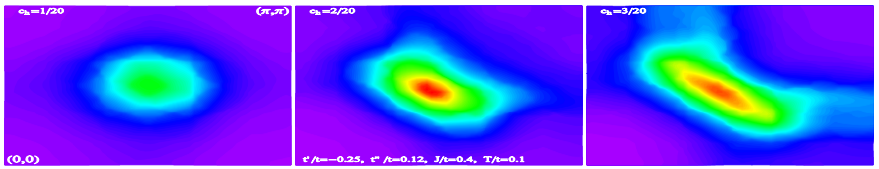

pseudogap
A
0.2
0.1
0
(a)
1
k_x/π
−8
−4
0
ω/t
4
0.5

$c_h = 0.05$


A
0.2
0.1
0
(c)
1
k_x/π
−8
−4
0
ω/t
4
0.3

$c_h = 0.17$

Fermi surface evolution: A(k,ω=0)



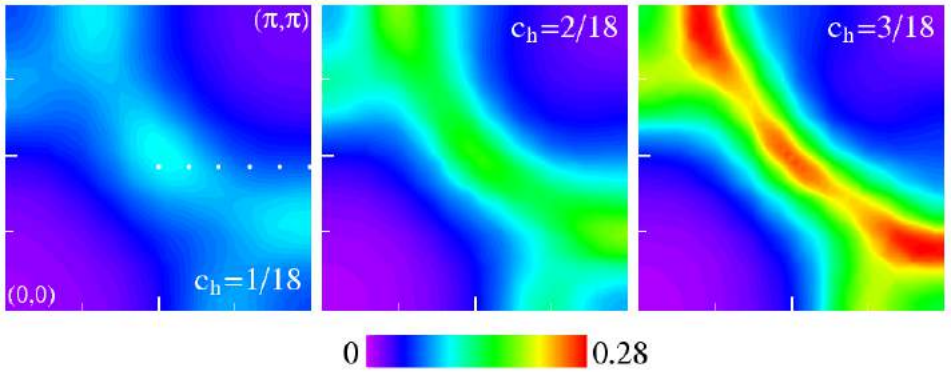
t - t'- t''- J model:

t'= - 0.3 t, t''=0.12 t,
J=0.4 t

c_h = 1/20, 2/20, 3/20

pseudogap

Fermi arc

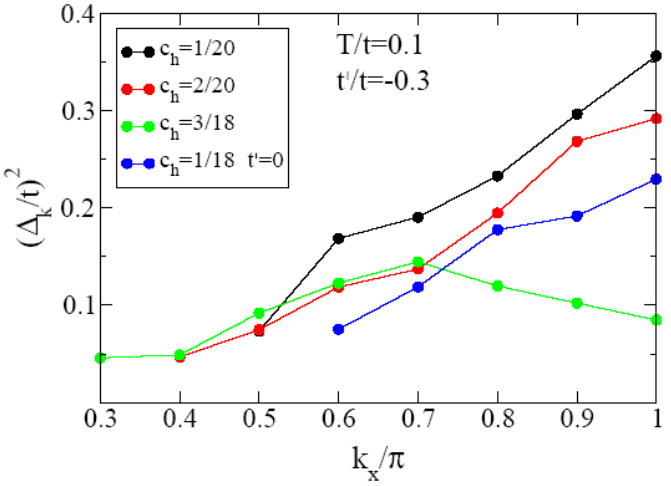


t - J model:

J=0.3 t

c_h = 1/18, 2/18, 3/18

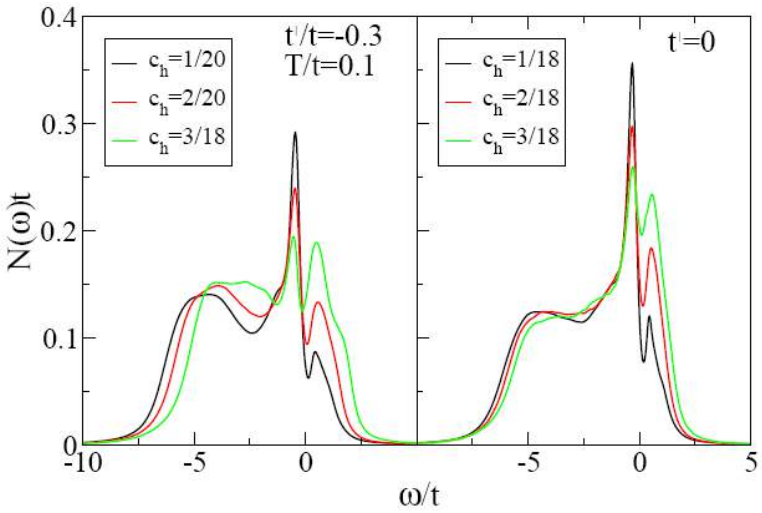
Pseudogap evolution:



pseudogap large:

a) antinodal region

b) low doping



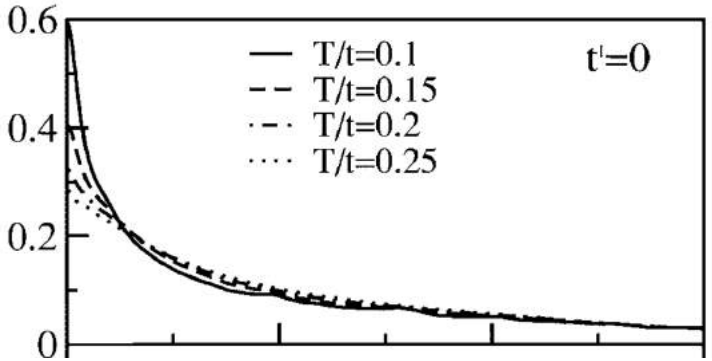
density of states:

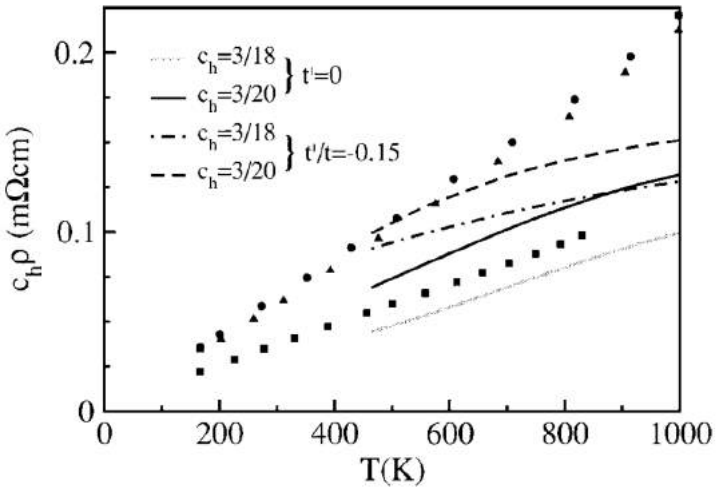
integrated pseudogap

t – J model: transport properties

dynamical (optical) conductivity:

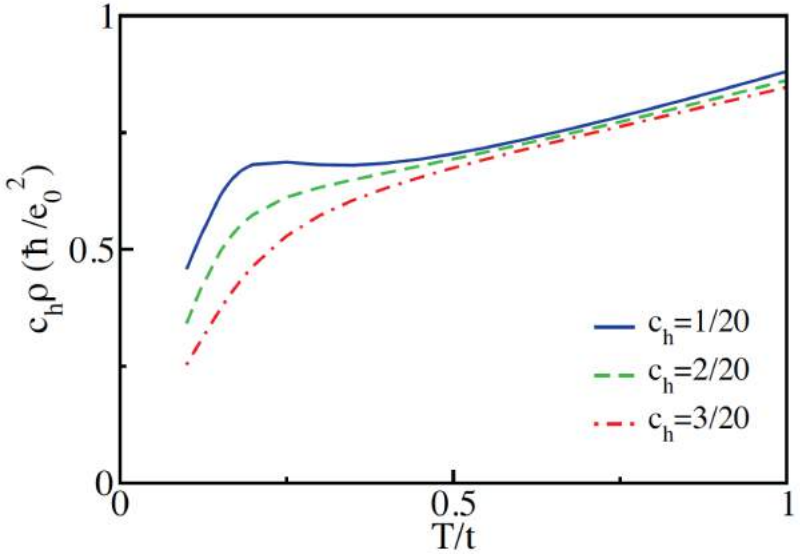
$$\sigma(\omega) = \frac{1 - \mathrm{e}^{-\beta\omega}}{\omega N} \mathrm{Re} \int_0^\infty dt \mathrm{e}^{i\omega t} \langle j_\alpha(t) j_\alpha \rangle$$





$$\sigma(\omega \to 0) = \sigma_0 = 1/\rho_0$$

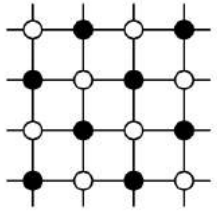
d.c. conductivity, resistivity



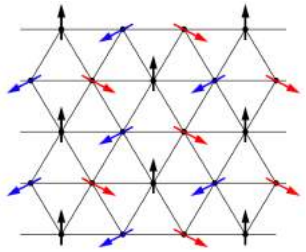
theory (FTLM): experiment

FTLM one of few methods for d.c. transport !

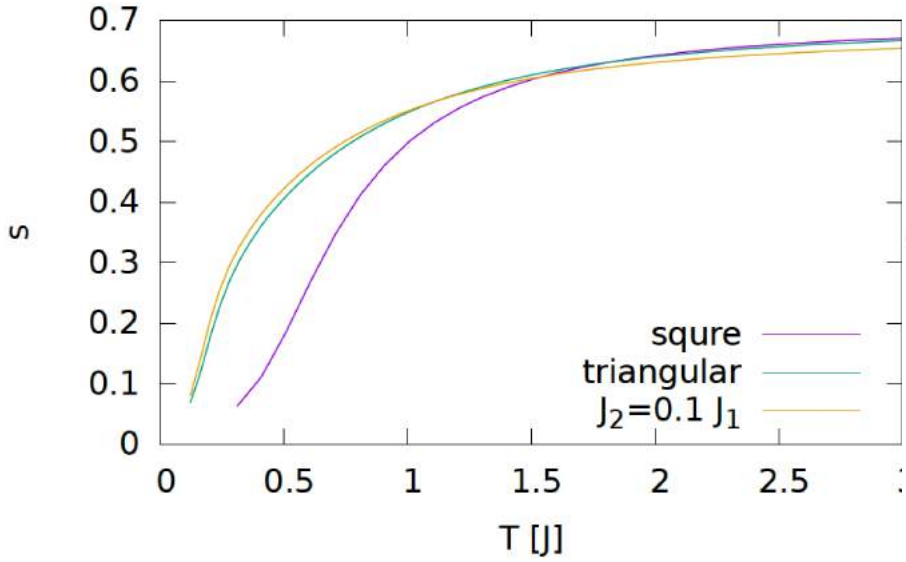
2D Heisenberg model: unfrustrated vs. frustrated (2017)



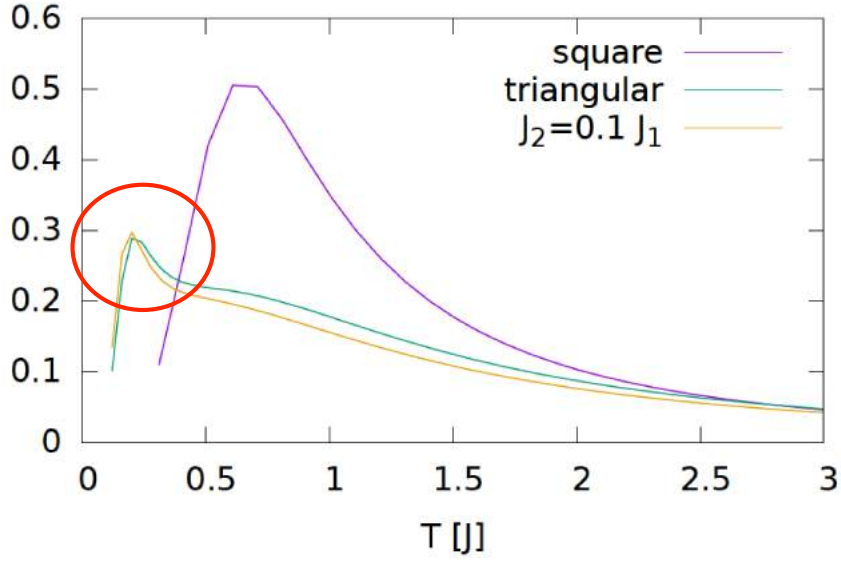
$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$



square lattice vs. triangular lattice



entropy



specific heat

Many-body localization: goal

Create a macroscopic quantum MB system which does not thermalize at any temperature and retains the information locally ?

Ideal noneqilibrium system = absence of thermalization

no d.c. transport at any T !

nonergodicity of (all) correlations

local quantitites : qubits

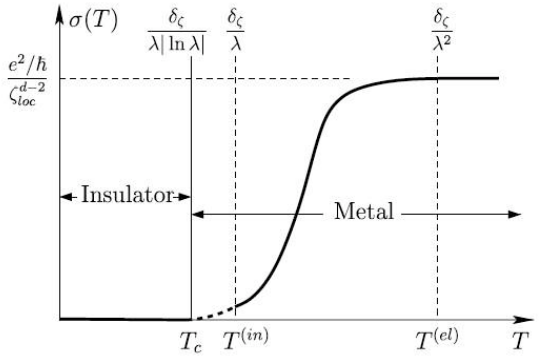
no leakage of quantum information

Numerical challenge: T >> 0 + low-ω (long t) dynamics !

What is MBL and why is it so interesting ?

Nonergodic behaviour in a macroscopic MB quantum system: T > 0

- non-interacting (NI) fermions on disordered lattice: Anderson localization
- integrable MB models: Heisenberg chain etc…
- systems undergoing phase transition (macroscopic ordering at $T < T_c$)



systems = correlations + large disorder ?

Basko, Aleiner, Altshuler (2006):

-MI transition at T=T* at fixed disorder W

-MI transition at $W=W_c$ even at $T = \infty$!

> 700 theoretical papers after 2006 > 100 papers / year

Does MBL exist (phase transition or crossover ..) ?

Which are properties of the ergodic and non-ergodic phase ?

'Standard' model of many-body localization

1D isotropic (or anisotropic) Heisenberg model + random fields:

$$H = J \sum_i \left[\frac{1}{2}(S^+_{i+1} S^-_i + S^-_{i+1} S^+_i) + \Delta S^z_{i+1} S^z_i \right] + \sum_i h_i S^z_i \qquad h_i \in [-W, W]$$

Jordan – Wigner transformation (1D)

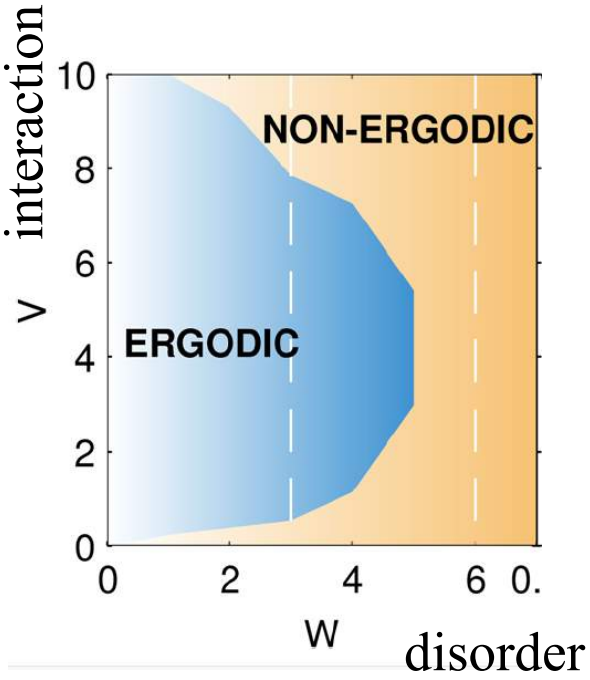
equivalent to disordered chain of interacting spinless fermions

= Anderson model + interaction

$$t = J/2$$

$$H = -t \sum_i (c^\dagger_{i+1} c_i + \text{h.c.}) + \sum_i h_i n_i + V \sum_i n_{i+1} n_i \qquad V = J\Delta$$

T ~ ∞: **phase diagram** (approximate ?)



$$H = -t\sum_i (c^\dagger_{i+1}c_i + \text{h.c.}) + \sum_i h_i n_i + V\sum_i n_{i+1}n_i$$

Bar Lev et al, PRL (2015)

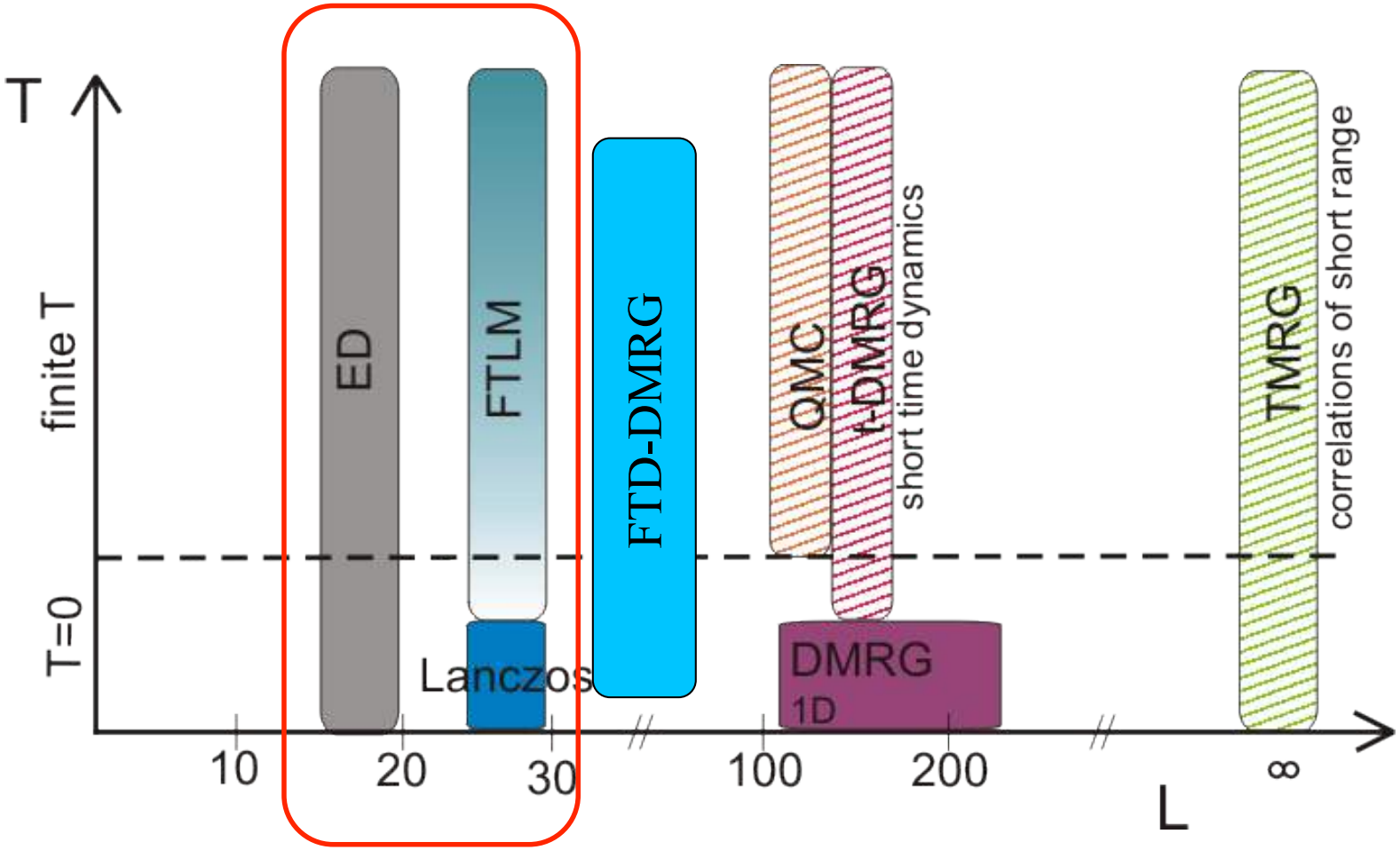
ergodic phase: $W < W_c(V)$

nonergodic (MBL) phase: $W > W_c(V)$

$W > W_c$:

- Poisson MB level statistics
- vanishing d.c. transport – spin (particle) , energy
- area (log) law for entanglement entropy increase
- non-ergodic behaviour of (all) correlation functions, no thermalization
- **local integrals of motion**

Numerical methods for MBL (dynamics)



MBL numerical problem: **T >> 0 + very long times - low ω !**

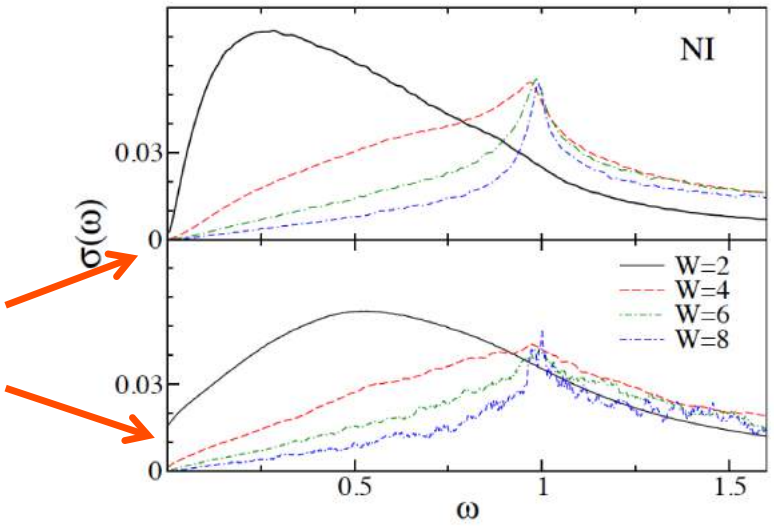
\+ large sizes ?

Characteristic feature: dynamical conductivity and d.c. transport

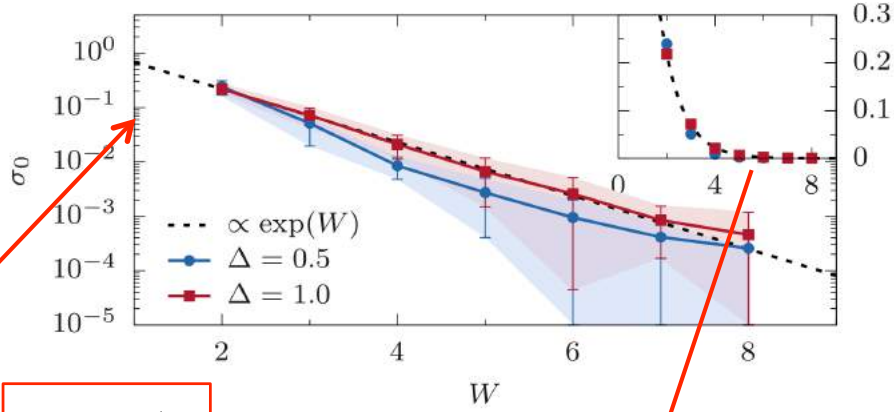
Barišić et al, PRB (2010, 2016)

vanishing d.c. transport σ_0 for $W > W_c \sim 5$

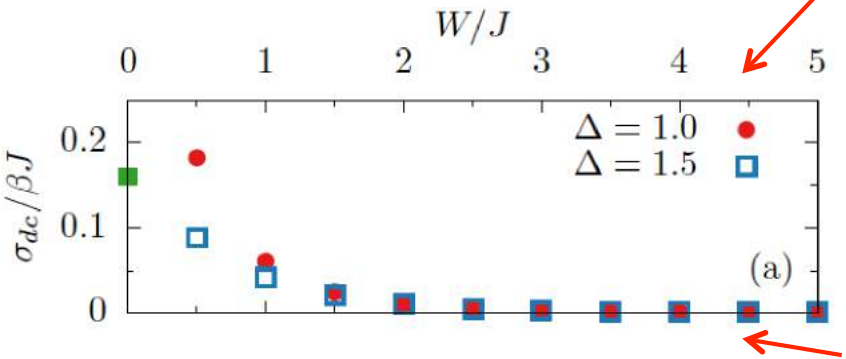
$$\sigma(\omega) \sim \sigma_0 + \zeta|\omega|^{\alpha} \qquad \alpha \sim 1$$



log !



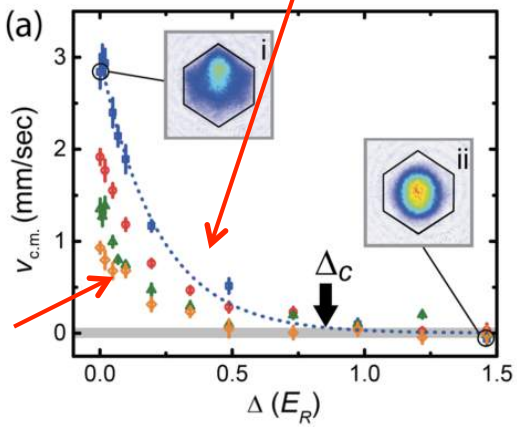
T >> 1:



sharp transition :

crossover ??

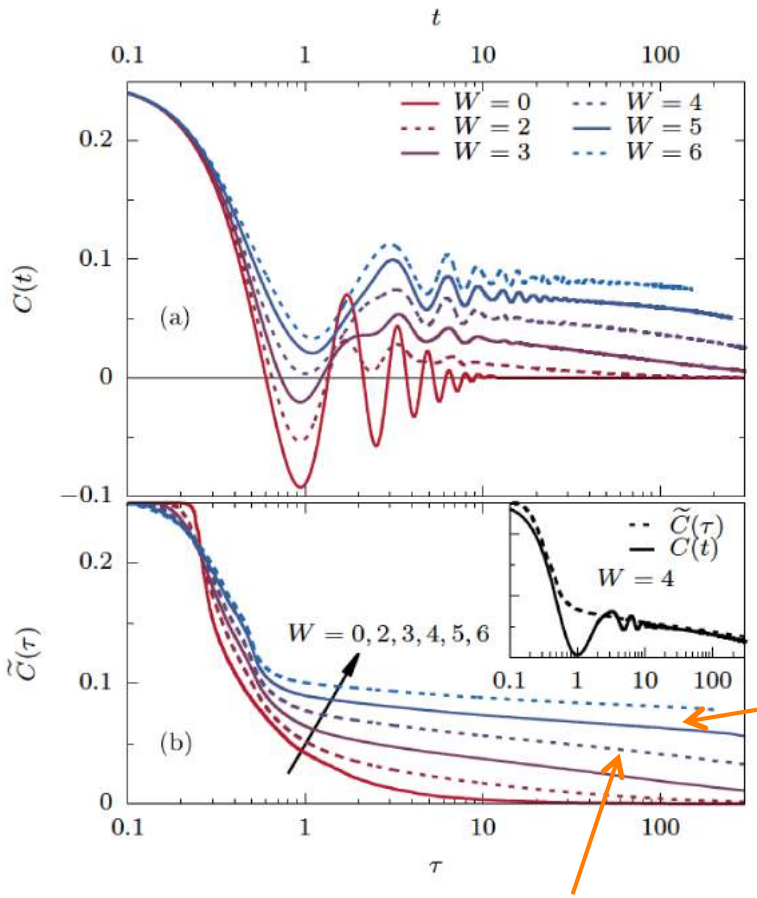
theory : experiment



Steinigeweg et al, PRB (2016)

Characteristic feature: nonergodicity and universal dynamics

Mierzejewski et al., PRB (2016)



$\tilde{C}(\tau) \sim \log \tau/\tau^*$

density-wave (imbalance) correlation function: $T = \infty$, $V = t$ (Δ=0.5), ED, L = 16

$$C(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\imath\omega^+ t}\langle n_{q=\pi}(t)\, n^\dagger_{q=\pi}\rangle$$

a) real-time dynamics: $C(t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-\imath\omega t} C(\omega)$

oscillations emerging from NI physics

b) 'quasi'-time dynamics: $\tilde{C}(\tau) = \int_{-1/\tau}^{1/\tau} \mathrm{d}\omega\, C(\omega)$

the same long-time variation

$$\tilde{C}(\tau) \sim C_0 + a\, t^{-\gamma}$$

nonergodic (MBL) phase: $W > W^* \sim 4$

$C_0 = C(t{=}\infty) > 0$ + anomalous time dependence $C(\omega) = C_0\delta(\omega) + C_{\mathrm{reg}}(\omega)$

Many open question and numerical challenges !