

17 Algebraic Methods in Many-Body Physics

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1 The many languages of nature

One of the most challenging and interesting phenomena in modern condensed matter physics is the one emerging from competing interactions in strongly correlated systems. The multiplicity of distinct and exotic quantum phases observed experimentally confronts us with new paradigms that question our understanding of the fundamental organizing principles behind such emergent phenomena [1]. The notion of symmetry has shaped our current conception of nature; however, nature is also full of broken symmetries. Thus, understanding the idea of invariance and its corresponding conservation laws is as fundamental as determining the causes that prevent such harmony, and leads to more complex behavior. While group theory and geometry have been fundamental to the physics of the Twentieth Century, only recently has topology become central to the nascent field of topological quantum matter [2–4]. Our ultimate goal is to exploit and extend tools borrowed from these fields of mathematics to unveil and master those underlying organizing principles [5]. These principles may lead the way to designing new materials and devices with specific functionalities and unprecedented technological applications.

On general grounds, the nature and degree of entanglement of the quantum state, characterizing the various thermodynamic phases of matter, is at the core of such complex behavior [6]. For instance, the last three decades have witnessed the discovery of fractional charges and skyrmion excitations in quantum Hall liquids [7] (i.e., electrons confined to two space dimensions in the presence of strong external magnetic fields). Another example of current interest is provided by unconventional high-temperature superconductivity and its startling strange metallic behavior [8]. From the theoretical viewpoint the hurdle is in the presence of non-linear couplings, non-perturbative phenomena, and a panoply of competing quantum orders. These systems happen to be strongly correlated because no obvious small coupling constant exists, and consequently exhibit high sensitivity to small parameter changes. The importance of developing a methodology based on qualitatively new concepts going beyond traditional mean-field (MF) and semi-classical approximations, that treats all possible competing orders on an equal footing with no privileged fixed-point phenomenon, becomes manifest. Despite great advances, there is a lack of a systematic and reliable methodology to study and predict the behavior of these complex systems. It is the purpose of these lecture notes to present some steps in that direction. Describing the structure and behavior of matter entails studying systems of interacting quantum constituents (bosons, fermions, gauge fields, spins). In the quantum-mechanical description of matter, each physical system is naturally associated with a *language* of operators (for example, quantum spin-1/2 operators) and thus to an algebra realizing this language (e.g., the Pauli spin algebra generated by a family of commuting quantum spin-1/2 operators). It is our point of view that crucial to the successful understanding of the mechanisms driving complexity is the realization of *dictionaries* (isomorphisms) connecting the different languages of nature and therefore linking seemingly unrelated physical phenomena [9, 10]. The existence of dictionaries provides not only a common ground to explore complexity but leads naturally to the fundamental concept of *universality*, meaning that different physical systems show the same behavior. In this way, there is a concept of physical equivalence hidden in these dictionaries.

On the other hand, the notion of algebra and its homomorphisms have also been essential to unravel hidden structures in theoretical physics: Internal symmetries which are hidden in a given algebraic representation of a model become manifest in another representation. In 1928 Jordan and Wigner [11] made a first step relating quantum spin $S = 1/2$ degrees of freedom to particles with fermion statistics. A simple application of their contribution is the mapping between the isotropic XY chain describing quantum magnets and the tight-binding spinless fermion model, which can be exactly solved in one spatial dimension. From the group theoretical viewpoint, an internal $U(1)$ continuous symmetry (related to particle number conservation), for instance, is evidenced in the fermion representation of the XY model which was hidden in the spin representation. Overall, what Jordan and Wigner established was an isomorphism of $*$ -algebras, i.e., an isomorphism between the Pauli and fermion algebras [12].

In this chapter we present a unifying algebraic framework for interacting quantum systems. We show that exact algebraic and group theoretic methods are one of the most elegant and promising approaches towards a complete understanding of quantum phases of matter and their corresponding phase transitions. Can we connect the different (spin-particle-gauge) languages of nature within a single algebraic framework? We will present a fundamental theorem which connects operators generating different algebras (e.g., $su(D)$ spin-particle connections), unifying the different languages known so far in the quantum-mechanical description of matter. We will illustrate the methodology with various examples borrowed from strongly correlated physics, including quantum magnets and superfluids. Applications aim at:

- Illustrating connections between seemingly unrelated physical phenomena
- Unveiling hidden symmetries to identify new states of matter
- Identifying order parameters (OPs) in phase transitions [13, 14]
- Establishing exact (algebraic) solutions and developing better approximation schemes
- Explaining the use of languages in quantum simulations

The chapter has been written with the intention of providing the reader with the most fundamental concepts involved in our algebraic framework and how they apply to study complex phenomena. Much more details and examples can be found in the original manuscripts [9, 10, 12, 15].

2 Algebraic approach to interacting quantum systems

The theory of operator algebras on Hilbert spaces was initiated by Murray and von Neumann [16] as a tool to study unitary representations of groups, and as a framework for a reformulation of quantum mechanics. This area of research continued its development independently in the realm of mathematical physics, and therefore knowledge of those investigations remained bounded to specialists. For use of C^* and W^* algebras as a framework for quantum statistical mechanics one can look at the books of Bratteli and Robinson [17]. For the purposes of our presentation one only needs to have an elementary background in basic algebra [18], and specially group theory [19], in particular, Lie algebras and groups.

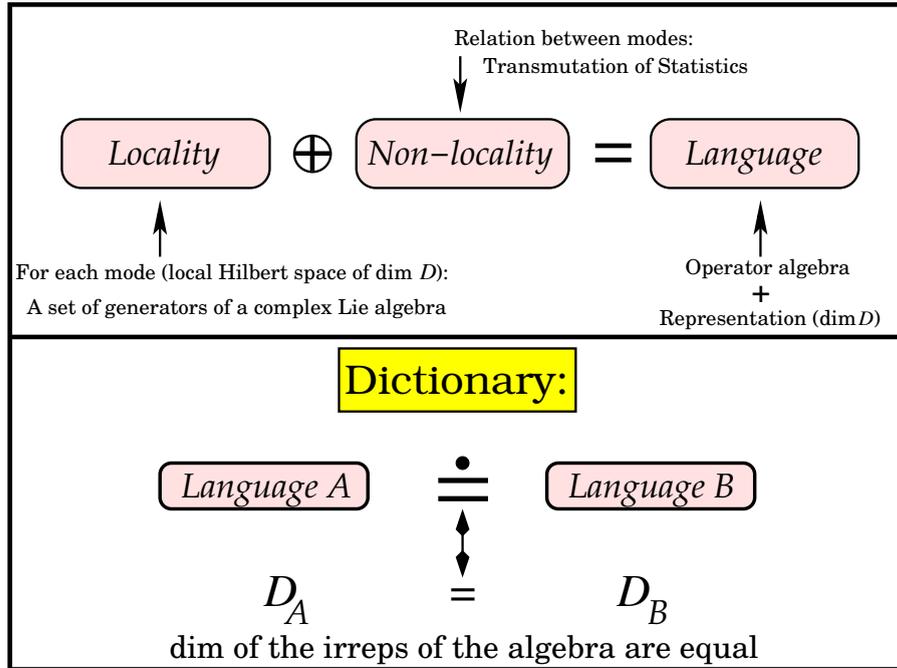


Fig. 1: Definition of a language and fundamental theorem [12, 9] behind the construction of the dictionaries of nature. In the upper panel we show schematically what elements define a language $= \mathcal{A} \wedge \Gamma_A$, where \mathcal{A} is the algebra and Γ_A a particular irrep. In the lower panel we establish the criteria to build a dictionary, given two languages A and B . This criteria is based upon Burnside's theorem of algebra [18].

Here we are concerned with quantum lattice systems. A quantum lattice is identified with \mathbb{Z}^{N_s} , where N_s is the total number of lattice sites (or modes). Associated to each lattice site $\mathbf{i} \in \mathbb{Z}^{N_s}$ there is a Hilbert space \mathcal{H}_i of finite dimension D describing the “local” modes. The total Hilbert space is $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ (we will also consider its symmetric and antisymmetric subspaces). Since we are mostly interested in zero temperature properties, a state of the system is simply a vector $|\Psi\rangle$ in \mathcal{H} , and an observable is a self-adjoint operator $\mathcal{O} : \mathcal{H} \rightarrow \mathcal{H}$. The dynamical evolution of the system is determined by its Hamiltonian H . The topology of the lattice, dictated by the connectivity and range of the interactions in H , is an important element in establishing complexity. In the case of quantum continuous systems we can still use the present formalism after discretizing space. Going beyond this approach is outside the scope of these notes.

As mentioned above, each physical system is naturally associated with a language of operators, and thus to an algebra realizing this language. Formally, a *language* is defined by an operator algebra and a specific representation of the algebra. We use the following notation: $language = \mathcal{A} \wedge \Gamma_A$, where \mathcal{A} is the operator algebra and Γ_A is a particular irreducible representation (irrep) of the local algebra \mathcal{A}_j associated to \mathcal{A} , of dimension $\dim \Gamma_A = D$ (see Fig. 1).

For the sake of clarity, let us choose the phenomenon of magnetism to illustrate the key ideas. This is one of the most intriguing and not fully understood problems in condensed matter physics where strong correlations between electrons (of electrostatic origin) are believed to be the essence of the problem. To describe the phenomenon using a minimal model (i.e., a

model that only includes the relevant degrees of freedom) distinct approaches can be advocated depending upon the itineracy of the electrons that participate in the magnetic processes. In one extreme (e.g., insulators) a description in terms of localized quantum spins is appropriate, while in the other (e.g., metals) delocalization of the electrons is decisive and cannot be ignored. We immediately identify the languages associated to each description: quantum spins (e.g., Pauli algebra) and fermions (spin-1/2 Fermi algebra). Are these really different descriptions? Is there a dictionary that may connect the two languages? Let's assume that we decide to use the quantum spins language. What other seemingly unrelated phenomena are connected to magnetism? Can we relate phases of matter corresponding to dissimilar phenomena? Can an arbitrary physical system be mapped, for instance, onto a pure magnetic system (an array of quantum spins)? In the following we will answer these questions by examples. As mentioned above, a fundamental concept of universality, complementary to the one used in critical phenomena, emerges as a consequence of unveiling the hidden unity in the quantum-mechanical description of matter.

3 Bosonic and hierarchical languages

A *bosonic language* is a set of operators grouped in subsets S_i (associated to each mode) that satisfy the conditions

- Each element \mathfrak{b}_i^μ of S_i ($\mu \in [1, N_g]$) belongs to the algebra of endomorphisms for the vector space \mathcal{H}_i over the field of complex numbers \mathbb{C} , $\mathfrak{b}_i^\mu : \mathcal{H}_i \rightarrow \mathcal{H}_i$, and these elements are linearly independent.
- The elements of S_i generate a monoid [20] of linear transformations under the associative product in the algebra which acts irreducibly on \mathcal{H}_i in the sense that the only subspaces stabilized by S_i are \mathcal{H}_i and $\mathbf{0}$ ($\mathbf{0}$ is the null vector).
- If \mathfrak{b}_i^μ and \mathfrak{b}_j^ν are elements of different subsets S_i and S_j , then $\mathfrak{b}_i^\mu \mathfrak{b}_j^\nu = \square(\mathfrak{b}_i^\mu, \mathfrak{b}_j^\nu) = \square(\mathfrak{b}_j^\nu, \mathfrak{b}_i^\mu)$.

Combining the associative product \square and the additive operations, we can define the non-associative Lie product $[\ , \]$, which is called commutator

$$[\mathfrak{b}_i^\mu, \mathfrak{b}_j^\nu] = \mathfrak{b}_i^\mu \mathfrak{b}_j^\nu - \mathfrak{b}_j^\nu \mathfrak{b}_i^\mu. \quad (1)$$

Using this product the last condition can be reformulated

$$[\mathfrak{b}_i^\mu, \mathfrak{b}_j^\nu] = 0, \quad \text{if } i \neq j. \quad (2)$$

The set S_i is not necessarily closed under the regular product (composition) or the Lie product (commutator). If the set S_i is closed under the Lie product

$$[\mathfrak{b}_i^\mu, \mathfrak{b}_i^\nu] = \sum_{\mu'=1}^{N_g} \lambda_{\mu'} \mathfrak{b}_i^{\mu'}, \quad \lambda_{\mu'} \in \mathbb{C}, \quad (3)$$

the elements of \mathcal{S}_i generate a Lie algebra \mathcal{S}_i . In addition, since each generator is represented by an endomorphism of \mathcal{H}_i there is a particular representation $\Gamma_{\mathcal{S}}$ of \mathcal{S}_i associated to the bosonic language. The second condition for a bosonic language implies that $\Gamma_{\mathcal{S}}$ is irreducible. The third condition implies that the global Lie algebra associated to the complete set of generators is the direct sum of local algebras \mathcal{S}_i , $\mathcal{S} = \bigoplus_i \mathcal{S}_i$. Therefore, if the set \mathcal{S}_i is closed under the Lie product, we can represent the bosonic language by the conjunction of the Lie algebra \mathcal{S} , and the irreducible representation $\Gamma_{\mathcal{S}}$: $\mathcal{S} \wedge \Gamma_{\mathcal{S}}$. The dimension of $\Gamma_{\mathcal{S}}$ is equal to the dimension of the local Hilbert space \mathcal{H}_i : $\dim \Gamma_{\mathcal{S}} = D$.

3.1 Languages and dictionaries: a fundamental theorem

The demonstration of the fundamental theorem of this section is a direct consequence of the classical theorem of Burnside [18]

Burnside's theorem. Let G be a monoid of linear transformations in a finite dimensional vector space \mathbf{V} over an algebraically closed field F (in quantum mechanics, the complex numbers \mathbb{C}), that acts irreducibly on \mathbf{V} in the sense that the only subspaces stabilized by G are \mathbf{V} and $\mathbf{0}$. Then G contains a base for $\text{End}_F \mathbf{V}$ (ring of endomorphisms of \mathbf{V} over F).

The following theorem shows that two languages are *equivalent* if they have in common the dimension D of their local Hilbert space \mathcal{H}_i , and D is finite.

Fundamental Theorem [12, 9]: Given two bosonic languages having the same finite dimension D of their local Hilbert spaces \mathcal{H}_i , the generators of one of them can be written as a polynomial function of the generators of the other language and vice versa.

This theorem provides the necessary and sufficient conditions to connect two bosonic languages. What is the unifying concept behind the construction of the dictionaries (isomorphisms) of nature? When is it possible to build a dictionary between two arbitrary (bosonic, fermionic or anyonic) languages? The answers lie in the application of the fundamental theorem together with the transmutation of statistics [9] (see Fig. 1). This will be done in Section 4.

The fundamental theorem establishes the existence of dictionaries connecting languages within the same class. As a consequence, we can use any bosonic language in the class to describe a given physical phenomena. The natural question which emerges from this result is: What is the most appropriate language in a given class for describing our particular problem? There is no generic answer to this question. Nonetheless, the following two corollaries give an important hint for problems which are invariant under particular transformations, because they relate the notion of language to the generators of symmetry groups.

- **Corollary I:** In each class of bosonic languages there is at least one which is the conjunction of a Lie algebra \mathcal{S} and an irreducible representation $\Gamma_{\mathcal{S}}$ ($\mathcal{S} \wedge \Gamma_{\mathcal{S}}$), i.e., the generators of the bosonic language are generators of the Lie algebra \mathcal{S}_i in the rep. $\Gamma_{\mathcal{S}}$.

Consider the Lie algebra $\mathcal{L}_i = u(1) \oplus su(2)$ ($\mathcal{S} = \bigoplus_i \mathcal{L}_i$) whose generators are $\{I_i, S_i^x, S_i^y, S_i^z\}$. The three basis elements S_i^α (linear and Hermitian operators) of the Lie algebra $su(2)$ for each

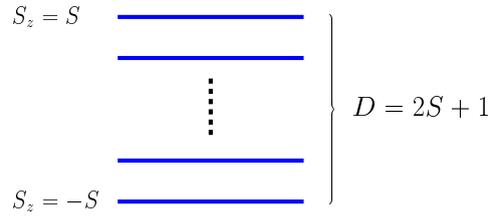


Fig. 2: Spin S representation of $su(2)$ of dimensionality $D = 2S + 1$.

lattice site (mode) \mathbf{i} ($\mathbf{i} = 1, \dots, N_s$) satisfy the equal-time commutation relations

$$[S_{\mathbf{i}}^a, S_{\mathbf{j}}^b] = i\delta_{\mathbf{ij}} \epsilon_{abc} S_{\mathbf{i}}^c, \quad a, b, c = x, y, z, \quad (4)$$

with ϵ the totally antisymmetric Levi-Civita symbol. Equivalently, in terms of the ladder operators $S_{\mathbf{j}}^{\pm} = S_{\mathbf{j}}^x \pm iS_{\mathbf{j}}^y$

$$[S_{\mathbf{j}}^+, S_{\mathbf{j}}^-] = 2S_{\mathbf{j}}^z, \quad [S_{\mathbf{j}}^z, S_{\mathbf{j}}^{\pm}] = \pm S_{\mathbf{j}}^{\pm}, \quad \{S_{\mathbf{j}}^+, S_{\mathbf{j}}^-\} = 2(S(S+1) - (S_{\mathbf{j}}^z)^2), \quad (5)$$

where $D = 2S + 1$ is the dimension of the irreducible representation Γ_S^D (Fig. 2). The demonstration of corollary I shows that any bosonic problem with a local Hilbert space of dimension D can be described with $su(2)$ -spins of magnitude (representation) $S = (D - 1)/2$.

3.2 Hierarchical language

A given bosonic language will be called *hierarchical* if any local physical operator $\hat{O}_{\mathbf{i}}: \mathcal{H}_{\mathbf{i}} \rightarrow \mathcal{H}_{\mathbf{i}}$, can be written as a linear combination of the generators of the language

$$\hat{O}_{\mathbf{i}} = \sum_{\mu=1}^{N_g} \lambda_{\mu} \mathbf{b}_{\mathbf{i}}^{\mu}, \quad \lambda_{\mu} \in \mathbb{C}. \quad (6)$$

- **Corollary II:** In each class of bosonic languages there is at least one which is hierarchical, and its generators are the identity and the generators of $su(N=D)$ in the fundamental representation.

Proof: The Lie algebra generated by the identity $I_{\mathbf{i}}$ and the generators of $su(N)$ is $\mathcal{L}_{\mathbf{i}} = u(1) \oplus su(N)$. Since $\dim \mathcal{L}_{\mathbf{i}} = N^2$, the generators of $\mathcal{L}_{\mathbf{i}}$ are also a base for $\text{End}_{\mathbb{C}} \mathcal{H}_{\mathbf{i}}$ if $D = \dim \mathcal{H}_{\mathbf{i}} = N$.

A consequence of corollary II is that the generators of any language can be expressed as a linear combination of generators of a hierarchical language (HL) in the same class. The most trivial example is given by the class of bosonic languages with $D = 2$. The generators of any language (any two level system) in the same class can be written as a linear combination of the identity and the Pauli matrices. We will see later that corollary II is the key to get a hierarchical classification of the possible broken symmetries of a given physical Hamiltonian.

3.2.1 Example: different classes of bosonic particles

Canonical bosons with N_f different flavors α satisfy the standard canonical commutation relations

$$\begin{cases} [b_{i\alpha}, b_{j\beta}] = [b_{i\alpha}^\dagger, b_{j\beta}^\dagger] = 0, \\ [b_{i\beta}, b_{j\alpha}^\dagger] = \delta_{ij} \delta_{\alpha\beta}, [n_{i\alpha}, b_{j\beta}^\dagger] = \delta_{ij} \delta_{\alpha\beta} b_{i\alpha}^\dagger. \end{cases} \quad (7)$$

The local Hilbert space is infinite dimensional. Consider now two additional examples that illustrate in detail the contents of the fundamental theorem and the subsequent corollaries. The first example corresponds to hard-core (HC) bosons with N_f different flavors α . Since they are HC, only single occupancy is allowed, i.e., the eigenvalues of $\bar{n}_i = \sum_\alpha \bar{n}_{i\alpha}$ are either 0 or 1 ($(\bar{b}_{i\alpha}^\dagger)^2 = 0$, and $\bar{n}_{i\alpha} = \bar{b}_{i\alpha}^\dagger \bar{b}_{i\alpha}$ is the number operator for the flavor α at the site i). The minimal set S'_i of operators that we can use to generate a bosonic language which is appropriate for HC bosons is: $S'_i = \{\mathbf{1}_i, \bar{b}_{i\alpha}^\dagger, \bar{b}_{i\alpha}\}$ with $1 \leq \alpha \leq N_f$. It can be shown that this set satisfies the three requirements for a bosonic language. The dimension of the local Hilbert space for these endomorphisms is $D = N_f + 1$. Then by varying the total number of flavors we can generate all possible values of D . Since each class of bosonic languages is characterized by the value of D , these HC bosons provide an example of a bosonic language in each class. It is clear that the set S'_i is not closed under the Lie product. Therefore, we cannot associate a Lie algebra to this minimal bosonic language. However, if we increase the number of generators in the set S'_i by including bilinear forms of the type $\bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}$, then the new set $\mathcal{S}_i = \{\mathbf{1}_i, \bar{b}_{i\alpha}^\dagger, \bar{b}_{i\alpha}, \bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}\}$, with $1 \leq \alpha, \beta \leq N_f$, becomes closed under the Lie product

$$\begin{cases} [\bar{b}_{i\alpha}, \bar{b}_{j\beta}] = [\bar{b}_{i\alpha}^\dagger, \bar{b}_{j\beta}^\dagger] = 0, \\ [\bar{b}_{i\beta}, \bar{b}_{j\alpha}^\dagger] = \delta_{ij} (\delta_{\alpha\beta} - \bar{n}_i \delta_{\alpha\beta} - \bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}), \\ [\bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}, \bar{b}_{j\gamma}^\dagger] = \delta_{ij} \delta_{\beta\gamma} \bar{b}_{i\alpha}^\dagger. \end{cases} \quad (8)$$

This means that the extended set \mathcal{S}_i is now a set of generators for a Lie algebra in a particular representation. From the commutation relations (Eq. (8)) we can conclude that \mathcal{S}_i is the direct sum of an $u(1)$ algebra, generated by the identity $\mathbf{1}_i$, and an $su(N)$ ($N = D = N_f + 1$) algebra generated by $\{\bar{b}_{i\alpha}^\dagger, \bar{b}_{i\alpha}, \bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}\}$: $\mathcal{S}_i = u(1) \oplus su(N)$. The representation Γ_S is the fundamental representation of $su(N)$ ($\dim \Gamma_S = N$). Therefore, the new language is a hierarchical one.

We will see in later sections that *canonical fermions* can be transformed continuously into bosons which are HC in each flavor

$$\begin{cases} [\tilde{b}_{i\alpha}, \tilde{b}_{j\beta}] = [\tilde{b}_{i\alpha}^\dagger, \tilde{b}_{j\beta}^\dagger] = 0, \\ [\tilde{b}_{i\alpha}, \tilde{b}_{j\beta}^\dagger] = \delta_{ij} \delta_{\alpha\beta} (1 - 2\tilde{n}_\alpha), [\tilde{n}_{i\alpha}, \tilde{b}_{j\beta}^\dagger] = \delta_{ij} \delta_{\alpha\beta} \tilde{b}_{i\alpha}^\dagger, \end{cases} \quad (9)$$

which implies $\{\tilde{b}_{i\alpha}, \tilde{b}_{i\alpha}^\dagger\} = 1$. The Lie algebra generated by these bosonic operators is $\mathcal{L} = \bigoplus_{\alpha,i} su(2)$, i.e. each set $\{\tilde{b}_{i\alpha}^\dagger, \tilde{b}_{i\alpha}, \tilde{n}_{i\alpha} - 1/2\}$ generates an $su(2)$ algebra.

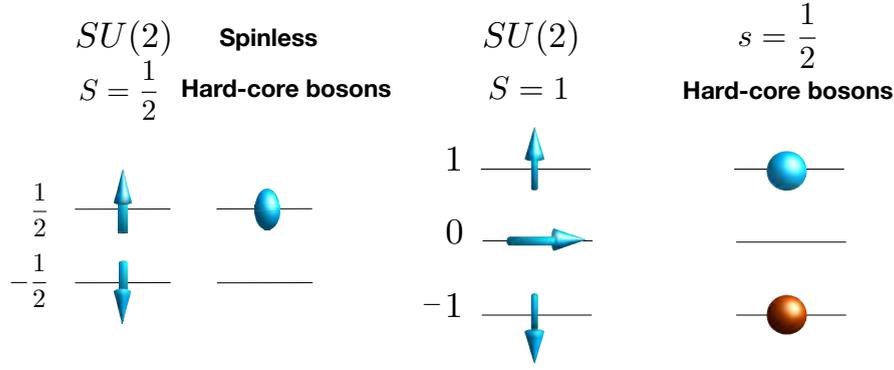


Fig. 3: Local Hilbert spaces \mathcal{H}_i , for the spin and HC bosonic worlds, at site (mode) i : (a) $S = \frac{1}{2}$ (dimension $D = 2$), (b) $S = 1$ (dimension $D = 3$).

3.2.2 Example: Matsubara-Matsuda transformation and its generalization to $S = 1$

To explain the peculiar properties of liquid Helium II, Matsubara and Matsuda [21] introduced a lattice model of atoms and holes. Think of a Helium-4 atom as a spinless HC boson ($N_f = 1$). On each lattice site, one can accommodate at most one Helium atom ($(\bar{b}_i^\dagger)^2 = 0$). Then, the simplest model of a liquid one can think of is described by a Hamiltonian with a kinetic energy term, with hopping amplitude t , and a nearest-neighbor density-density interaction of magnitude V . The expression for the Hamiltonian in terms of the first language defined by the sets S_i is

$$H_{\text{xxz}} = t \sum_{\langle i,j \rangle} (\bar{b}_i^\dagger \bar{b}_j + \bar{b}_j^\dagger \bar{b}_i) + V \sum_{\langle i,j \rangle} (\bar{n}_i - \frac{1}{2}) (\bar{n}_j - \frac{1}{2}), \quad (10)$$

where $\langle i, j \rangle$, refers to nearest-neighbors in an otherwise regular d -dimensional lattice. Since \bar{b}_i^\dagger , creating an atom at lattice point i , and \bar{b}_i , annihilating an atom at lattice point i , are not generators of a Lie algebra, the eventual global symmetries of H_{xxz} remain hidden in this particular language. However, if we *translate* H_{xxz} to the second $SU(2)$ -spin language using the dictionary provided by Matsubara and Matsuda [21]

$$S_i^+ = \bar{b}_i^\dagger, \quad S_i^- = \bar{b}_i, \quad S_i^z = \bar{b}_i^\dagger \bar{b}_i - \frac{1}{2} = \bar{n}_i - \frac{1}{2}, \quad (11)$$

we can immediately unveil the hidden symmetries of H_{xxz} . This dictionary maps single-site (mode) states in the following way (see Fig. 3)

$$|-\frac{1}{2}\rangle \longleftrightarrow |0\rangle, \quad |\frac{1}{2}\rangle \longleftrightarrow \bar{b}_i^\dagger |0\rangle, \quad (12)$$

where $|0\rangle$ is the vacuum of HC bosons. The well-known expression for H_{xxz} in terms of the $su(2)$ generators (i.e., the equivalent spin Hamiltonian) is

$$H_{\text{xxz}} = \sum_{\langle i,j \rangle} J_z S_i^z S_j^z + \frac{J_\perp}{2} (S_i^+ S_j^- + S_i^- S_j^+). \quad (13)$$

The magnetic couplings, J_z and J_\perp , are related to the original parameters, t and V , by the relations: $J_z = V$ and $J_\perp = 2t$. It is clear from Eq. (13) that the original model has a global

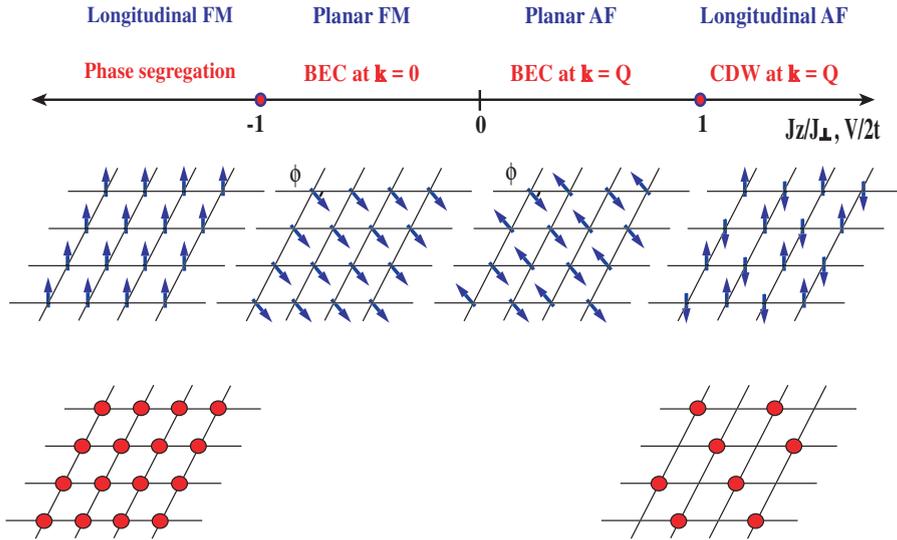


Fig. 4: Quantum phase diagram of the H_{xz} model Hamiltonian. \mathbf{Q} stands for the AF wave vector, and FM refers to the ferromagnetic phase.

$SU(2)$ invariance if $V = 2t$, i.e., it is in the isotropic Heisenberg point. The existence of this $SU(2)$ -symmetric point has a very important consequence for the quantum phase diagram (Fig. 4) of the bosonic model of Eq. (10): If there is a charge density wave (CDW) instability at that point, the $SU(2)$ invariance implies that *there is also* a Bose-Einstein (BE) condensation and vice versa. The order parameters (OPs) of both phases are different components of a unique OP in the spin language, i.e., the staggered magnetization of the antiferromagnetic (AF) phase ($t > 0$). The z -component of the staggered magnetization is mapped onto the CDW OP for the bosonic gas, while the transverse component is equivalent to the OP for the BE condensation. Only one of these two phases, which are coexisting at the $SU(2)$ invariant point, is stable when we depart from the symmetric point in any of both directions in parameter space (BE condensation if $V < 2t$ and CDW if $V > 2t$). In this very simple example we can see the advantages of using a HL ($su(2)$ in this case). In the first place, we can immediately recognize the high symmetry points. Secondly, we can describe an eventual broken symmetry state at those points in terms of a unified OP. If we were to use a non-HL to describe the same problem, we would find coexistence of more than one phase at the high symmetry points. The OPs of each of those phases are different components of the unified OP that we simply found with the HL. These ideas are developed in more detail in Section 5.4.

One can generalize [12] this idea to the case of HC bosons with two flavors, i.e., $N_f = 2$, that we can call spin $\sigma = \uparrow, \downarrow$, with the property that $\bar{b}_{i\sigma}^\dagger \bar{b}_{i\sigma'}^\dagger = 0$. In this case, the dictionary (isomorphic map) is given by (see Fig. 3, where $\uparrow(\downarrow)$ is represented by a blue(red) ball)

$$S_i^+ = \sqrt{2} (\bar{b}_{i\uparrow}^\dagger + \bar{b}_{i\downarrow}^\dagger), \quad S_i^- = \sqrt{2} (\bar{b}_{i\uparrow} + \bar{b}_{i\downarrow}), \quad S_i^z = \bar{n}_{i\uparrow} - \bar{n}_{i\downarrow}, \quad (14)$$

and single-site (mode) states map as

$$|-\mathbf{1}\rangle \longleftrightarrow \bar{b}_{i\downarrow}^\dagger |0\rangle, \quad |0\rangle \longleftrightarrow |0\rangle, \quad |\mathbf{1}\rangle \longleftrightarrow \bar{b}_{i\uparrow}^\dagger |0\rangle. \quad (15)$$

We will utilize this dictionary in later sections.

4 Transmutation of statistics: fermionic (anyonic) languages

We know that bosonic languages do not exhaust all possible languages of quantum mechanics. We have seen that the notion of bosonic languages is closely related to the concept of Lie algebras, since for each class of bosonic languages there is at least one language whose elements are generators of a Lie algebra. However, the same cannot be done for fermions. This can be easily seen by noticing that the condition for a bosonic language, Eq. (2), is not valid for fermions. In addition, the main consequence of Eq. (2) for a bosonic language is that the global Lie algebra is the direct sum of the local ones associated to each subset S_i . The generalization of these concepts to fermionic languages can be done by introducing the notion of Lie superalgebras (see for instance Ref. [19]), albeit we will not need this mathematical construct. The fermionic languages are associated to Lie superalgebras in the same way the bosonic languages are associated to Lie algebras. We will only consider the *fermionic language* generated by the *canonical* creation and annihilation operators

$$\begin{cases} \{c_{i\alpha}, c_{j\beta}\} = \{c_{i\alpha}^\dagger, c_{j\beta}^\dagger\} = 0, \\ \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij} \delta_{\alpha\beta}, \end{cases} \quad (16)$$

and other languages obtained by imposing particular local constraints on the canonical fermions. These generators, together with the identity, generate the Heisenberg Lie superalgebra. In analogy to the bosonic languages (see Eq. (2)), the Lie product (in the superalgebra) of two elements acting in different sites (different subsets S_i, S_j) is zero. Thus, instead of having a direct sum of local Lie algebras as in the bosonic case, we have a direct sum of local Lie superalgebras. In the case of canonical fermions the local Lie superalgebras are Heisenberg superalgebras.

4.1 Local and non-local transmutation of statistics

We will start by considering multiframe fermions $c_{i\alpha}^\dagger$ ($\alpha \in [1, N_f]$) which satisfy the canonical anticommutation relations (Eq. (16)). Other types of fermions, usually considered in physics, can be derived from the canonical ones by imposing particular constraints.

The canonical fermions can be transformed into bosons $\tilde{b}_{i\alpha}^\dagger$ which are hard-core in each flavor (the eigenvalues of $\tilde{n}_{i\alpha} = \tilde{b}_{i\alpha}^\dagger \tilde{b}_{i\alpha}$ are either 0 or 1, see Eq. (9)), i.e., two or more bosons of the same flavor are not allowed on the same site, but two of different flavors are permitted. In previous sections we have shown that a physical theory for objects obeying commutation relations (Lie brackets) can be formulated in terms of a bosonic language. By the present connection we will be able to extend this statement to fermions (or anyons, in general) through a transmutation of statistics. The local anticommutation relations are transmuted into anti-commutation relations (and vice versa) when the creation and annihilation operators are multiplied by the ‘‘local transmutator’’ $\hat{T}_{j\alpha}^\dagger$

$$c_{j\alpha}^\dagger = \tilde{b}_{j\alpha}^\dagger \hat{T}_{j\alpha}^\dagger, \quad (17)$$

where $\hat{T}_{j\alpha} = \exp(i\pi \sum_{\beta < \alpha} \tilde{n}_{j\beta})$ is the ‘‘local transmutator,’’ and we are assuming a particular

ordering for the flavor indices α . From the expression for $\hat{\mathcal{T}}_{j\alpha}$ it is clear that

$$\hat{\mathcal{T}}_{j\alpha}^2 = I, \quad \hat{\mathcal{T}}_{j\alpha}^\dagger = \hat{\mathcal{T}}_{j\alpha}. \quad (18)$$

In this way we have established a mapping between fermions and bosons which are operating locally (on a given orbital or mode j)

$$\mathcal{S}_j = \{\tilde{b}_{j\alpha}^\dagger, \tilde{b}_{j\alpha}, \tilde{n}_{j\alpha} - \frac{1}{2}\} \longleftrightarrow \hat{\mathcal{S}}_j = \{c_{j\alpha}^\dagger, c_{j\alpha}, \hat{n}_{j\alpha} - \frac{1}{2}\} \quad (19)$$

So far, we have only transmuted the commutation relations between generators which belong to the same site or subset \mathcal{S}_i . For commutation relations of two generators at different sites we need to introduce a non-local operator K_j . For spinless fermions, in one spatial dimension, Jordan and Wigner [11] introduced the so-called kink-operator

$$K_j^{1d} = \exp\left(i\pi \sum_{l < j} \hat{n}_l\right), \quad (20)$$

to establish a map between quantum $S = 1/2$ spins and spinless fermions

$$S_j^+ = c_j^\dagger K_j, \quad S_j^- = K_j^\dagger c_j, \quad S_j^z = \hat{n}_j - \frac{1}{2}, \quad (21)$$

where $K_j = K_j^{1d}$ for a one-dimensional lattice. The generalization of K_j to multiflavored canonical fermions is straightforward

$$K_j = \exp\left(i\pi \sum_{l < j, \alpha} \hat{n}_{l\alpha}\right). \quad (22)$$

The complete transmutator for canonical fermions $\mathcal{K}_{j\alpha}^\dagger$ is the product of the local and the non-local components

$$c_{j\alpha}^\dagger = \tilde{b}_{j\alpha}^\dagger \hat{\mathcal{T}}_{j\alpha}^\dagger K_j^\dagger = \tilde{b}_{j\alpha}^\dagger \mathcal{K}_{j\alpha}^\dagger. \quad (23)$$

Similarly, one can extend this idea of transmutation of statistics to particles satisfying general equal-time anyonic canonical commutation relations defined by an angle θ . To this end we need to generalize the transmutators to any statistical angle $0 \leq \theta \leq \pi$. By replacing the phase π by θ , $\mathcal{K}_{j\alpha}^\dagger(\theta)$ transmutes the bosons (or fermions) into (Abelian) anyons

$$\tilde{a}_{j\alpha}^\dagger = \tilde{b}_{j\alpha}^\dagger \hat{\mathcal{T}}_{j\alpha}^\dagger(\theta) K_j^\dagger(\theta) = \tilde{b}_{j\alpha}^\dagger \mathcal{K}_{j\alpha}^\dagger(\theta). \quad (24)$$

Like in the previous example, the local commutation relations are preserved ($\tilde{n}_{j\alpha} = \tilde{a}_{j\alpha}^\dagger \tilde{a}_{j\alpha}$, $\tilde{n}_j = \sum_{\alpha=1}^{N_f} \tilde{n}_{j\alpha}$)

$$\begin{cases} [\tilde{a}_{j\alpha}, \tilde{a}_{j\alpha}] = [\tilde{a}_{j\alpha}^\dagger, \tilde{a}_{j\alpha}^\dagger] = 0, \\ [\tilde{a}_{j\alpha}, \tilde{a}_{j\alpha}^\dagger] = 1 - 2\tilde{n}_{j\alpha}. \end{cases} \quad (25)$$

In this particular case, since there is a hard-core condition $\tilde{a}_{j\alpha}^\dagger \tilde{a}_{j\alpha}^\dagger = 0$, the operators also satisfy the following local anticommutation relations

$$\begin{cases} \{\tilde{a}_{j\alpha}, \tilde{a}_{j\alpha}\} = \{\tilde{a}_{j\alpha}^\dagger, \tilde{a}_{j\alpha}^\dagger\} = 0, \\ \{\tilde{a}_{j\alpha}, \tilde{a}_{j\alpha}^\dagger\} = 1. \end{cases} \quad (26)$$

Thus, the local anticommutation relations are also preserved under statistical transmutation. Clearly, Eq. (26) are the local anticommutation relations for canonical fermions. This is not surprising since the multiflavored hard-core bosons defined by Eq. (9) can be transmuted into canonical fermions (see Eq. (23)). For operators involving different indices we have to define an index ordering. For $(\mathbf{j}, \beta) > (\mathbf{i}, \alpha)$

$$\begin{cases} [\tilde{a}_{\mathbf{i}\alpha}, \tilde{a}_{\mathbf{j}\beta}]_\theta = [\tilde{a}_{\mathbf{i}\alpha}^\dagger, \tilde{a}_{\mathbf{j}\beta}^\dagger]_\theta = 0, \\ [\tilde{a}_{\mathbf{j}\beta}^\dagger, \tilde{a}_{\mathbf{i}\alpha}]_\theta = 0, \end{cases} \quad (27)$$

where $[A, B]_\theta = AB - \exp[i\theta]BA$. Even though $\mathcal{K}_{\mathbf{j}\alpha}^\dagger(\theta)$ is a non-local operator, it does not introduce long-range interactions if the model has only nearest-neighbor-range terms. The only consequence of changing the statistics of the particles is a change of the short-range interactions in the original basis. Therefore, the concept of particle statistics in one dimension becomes irrelevant since any physical system can be described with a bosonic language without changing the short-range character of the interactions.

There is another type of HC anyon that is relevant for the generalized Jordan-Wigner transformations [12] of Sections 4.3 and 5.1. They are defined as

$$\bar{a}_{\mathbf{j}\alpha}^\dagger = \bar{b}_{\mathbf{j}\alpha}^\dagger \hat{\mathcal{T}}_{\mathbf{j}\alpha}^\dagger(\theta) K_{\mathbf{j}}^\dagger(\theta) = \bar{b}_{\mathbf{j}\alpha}^\dagger \mathcal{K}_{\mathbf{j}\alpha}^\dagger(\theta), \quad (28)$$

with

$$\hat{\mathcal{T}}_{\mathbf{j}\alpha}(\theta) = \exp\left(i\theta \sum_{\beta < \alpha} \bar{n}_{\mathbf{j}\beta}\right) \quad \text{and} \quad K_{\mathbf{j}}(\theta) = \exp\left(i\theta \sum_{\mathbf{l} < \mathbf{j}, \alpha} \bar{n}_{\mathbf{l}, \alpha}\right). \quad (29)$$

Since the local commutation relations are preserved, we have $(\bar{n}_{\mathbf{j}\alpha} = \bar{a}_{\mathbf{j}\alpha}^\dagger \bar{a}_{\mathbf{j}\alpha}, \bar{n}_{\mathbf{j}} = \sum_{\alpha=1}^{N_f} \bar{n}_{\mathbf{j}\alpha})$

$$\begin{cases} [\bar{a}_{\mathbf{j}\alpha}, \bar{a}_{\mathbf{j}\alpha}] = [\bar{a}_{\mathbf{j}\alpha}^\dagger, \bar{a}_{\mathbf{j}\alpha}^\dagger] = 0, \\ [\bar{a}_{\mathbf{j}\alpha}, \bar{a}_{\mathbf{j}\alpha}^\dagger] = 1 - \bar{n}_{\mathbf{j}\alpha} - \bar{n}_{\mathbf{j}}. \end{cases} \quad (30)$$

Again, we need to define an index ordering for the deformed commutation relations involving operators with different indices. For $(\mathbf{j}, \beta) > (\mathbf{i}, \alpha)$

$$\begin{cases} [\bar{a}_{\mathbf{i}\alpha}, \bar{a}_{\mathbf{j}\beta}]_\theta = [\bar{a}_{\mathbf{i}\alpha}^\dagger, \bar{a}_{\mathbf{j}\beta}^\dagger]_\theta = 0, \\ [\bar{a}_{\mathbf{j}\beta}^\dagger, \bar{a}_{\mathbf{i}\alpha}]_\theta = 0. \end{cases} \quad (31)$$

In closing this section we would like to emphasize that these lecture notes deals with Abelian anyons only. The case of non-Abelian anyons is more subtle and beyond the scope of these lecture notes. In general, it is not clear how to realize non-Abelian representations of the braid group in terms of ‘‘Fock-space particles’’. We only know examples in terms of Weyl parafermions [22].

4.1.1 Example: Spin-1/2 – anyon mapping

The simplest case of a spin-anyon mapping that one can imagine is realized in the case $D = 2$. Consider the map ($\bar{n}_j = \bar{a}_j^\dagger \bar{a}_j$ and $0 \leq \theta < 2\pi$)

$$S_j^+ = \bar{a}_j^\dagger K_j(\theta), \quad S_j^- = K_j^\dagger(\theta) \bar{a}_j, \quad S_j^z = \bar{n}_j - \frac{1}{2}, \quad (32)$$

where the non-local statistical operator or transmutator $K_j(\theta)$ is given by

$$K_j(\theta) = \exp\left(i\theta \sum_{i < j} \bar{n}_i\right) = \prod_{i < j} (1 + (e^{i\theta} - 1) \bar{n}_i) \quad (33)$$

since $\bar{n}_j^2 = \bar{n}_j$ (for any θ), and satisfy $K_j(\theta)K_j^\dagger(\theta) = K_j^\dagger(\theta)K_j(\theta) = \mathbb{1}$. In this way we transformed the original localized spin $S = 1/2$ problem into an itinerant gas of (anyon) particles obeying the operator algebra (for $i \leq j$)

$$\begin{cases} [\bar{a}_i, \bar{a}_j]_\theta = [\bar{a}_i^\dagger, \bar{a}_j^\dagger]_\theta = 0, \\ [\bar{a}_i, \bar{a}_j^\dagger]_{-\theta} = \delta_{ij}(1 - (\exp[-i\theta] + 1)\bar{n}_j), \quad [\bar{n}_i, \bar{a}_j^\dagger] = \delta_{ij} \bar{a}_j^\dagger. \end{cases} \quad (34)$$

Each statistical angle θ provides a different particle language and defines the exchange statistics of the particles. The case $\theta = \pi$ corresponds to canonical spinless fermions [11] while $\theta = 0$ represents HC bosons [21]. In all cases one can accommodate up to a single particle ($p = 1$) per quantum state, $(\bar{a}_j^\dagger)^{p+1} = 0$ (i.e, the particles are HC). In order to construct a dictionary one also needs the inverse mapping

$$\begin{cases} \bar{a}_j^\dagger = \prod_{i < j} \left(\frac{e^{-i\theta} + 1}{2} + (e^{-i\theta} - 1) S_i^z \right) S_j^+ \\ \bar{a}_j = \prod_{i < j} \left(\frac{e^{+i\theta} + 1}{2} + (e^{+i\theta} - 1) S_i^z \right) S_j^- \\ \bar{n}_j = S_j^z + \frac{1}{2} \end{cases} \quad (35)$$

4.2 Transmutation of statistics in higher dimensions

We propose the following general expression for K_j

$$K_j = \exp\left(i \sum_{\mathbf{l}} \omega(\mathbf{l}, \mathbf{j}) \bar{n}_{\mathbf{l}}\right), \quad (36)$$

where $\omega(\mathbf{l}, \mathbf{j})$ is a function to be determined by imposing the transmutation of statistics. This is equivalent to the antisymmetric condition

$$e^{i\omega(\mathbf{l}, \mathbf{j})} = -e^{i\omega(\mathbf{j}, \mathbf{l})}, \quad \text{if } \mathbf{l} \neq \mathbf{j}, \quad \omega(\mathbf{l}, \mathbf{l}) = 0. \quad (37)$$

One possible solution is based on a generalized Chern-Simons construction for a $2d$ lattice: $\omega(\mathbf{l}, \mathbf{j}) = a(\mathbf{l}, \mathbf{j})$ (see Fig. 5). However, this is not the only possible solution. Another solution

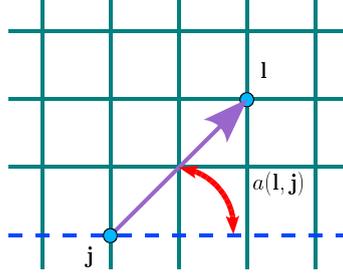


Fig. 5: One of the possible ways of defining the statistical transmutator K_j^{2d} in $2d$. The dotted line represents a fixed direction on the lattice.

in $2d$ is

$$\omega(\mathbf{l}, \mathbf{j}) = \pi(\Theta(j_1 - l_1)(1 - \delta_{l_1 j_1}) + \Theta(j_2 - l_2) \delta_{l_1 j_1}), \quad (38)$$

with $\mathbf{l} = l_1 \mathbf{e}_1 + l_2 \mathbf{e}_2$ and $\mathbf{j} = j_1 \mathbf{e}_1 + j_2 \mathbf{e}_2$. The advantage of this solution is that its generalization to higher dimensions is straightforward ($\mathbf{l} = \sum_a l_a \mathbf{e}_a$, and $a \in [1, d]$). For instance, in $3d$ we have

$$\omega(\mathbf{l}, \mathbf{j}) = \pi(\Theta(j_1 - l_1)(1 - \delta_{l_1 j_1}) + \Theta(j_2 - l_2) \delta_{l_1 j_1} (1 - \delta_{l_2 j_2}) + \Theta(j_3 - l_3) \delta_{l_1 j_1} \delta_{l_2 j_2}). \quad (39)$$

4.3 Jordan-Wigner transformation for $S = 1$

Mathematically, the Jordan-Wigner transformation [11] involves the $S = 1/2$ irreducible representation of the Lie group $SU(2)$. In this section we generalize this transformation to $S = 1$. As in the case of $S = 1/2$ HC bosons (Fig. 3) the dimension of the local Hilbert space is $D = 3$. In this case, the particle Hilbert space corresponds to spin- $\frac{1}{2}$ (two flavors) fermions with the constraint of no double occupancy. This constraint can be taken into account by introducing the Hubbard operators $\bar{c}_{j\alpha}^\dagger = c_{j\alpha}^\dagger (1 - n_{j\bar{\alpha}})$ and $\bar{c}_{j\alpha} = (1 - n_{j\bar{\alpha}}) c_{j\alpha}$ ($\alpha = 1, -1; \bar{\alpha} = -\alpha$), which form a subalgebra of the so-called double graded algebra $\text{Spl}(1,2)$ [19]. From Fig. 3 one realizes that S_j^z is the difference between the occupation numbers of the two different fermion flavors. S_j^+ must be a linear combination of annihilation and creation operators since we need to annihilate one fermion to go from $S_j^z = -1$ to the $S_j^z = 0$ state and to create the other fermion to go from $S_j^z = 0$ to $S_j^z = 1$. To simplify notation we introduce the following composite operators

$$f_j^\dagger = \bar{c}_{j1}^\dagger + \bar{c}_{j\bar{1}}, \quad f_j = \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger. \quad (40)$$

For spins on a lattice we again fermionize the spins and reproduce the correct spin algebra with the following transformation

$$S_j^+ = \sqrt{2} (\bar{c}_{j1}^\dagger K_j + K_j^\dagger \bar{c}_{j\bar{1}}), \quad S_j^- = \sqrt{2} (K_j^\dagger \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger K_j), \quad S_j^z = \bar{n}_{j1} - \bar{n}_{j\bar{1}}, \quad (41)$$

whose inverse manifests the nonlocal character of the mapping

$$f_j^\dagger = \frac{1}{\sqrt{2}} \exp\left(i\pi \sum_{i < j} (S_i^z)^2\right) S_j^+, \quad f_j = \frac{1}{\sqrt{2}} \exp\left(-i\pi \sum_{i < j} (S_i^z)^2\right) S_j^-, \quad (42)$$

$$\bar{c}_{j1}^\dagger = S_j^z f_j^\dagger, \quad \bar{c}_{j1} = f_j S_j^z, \quad \bar{c}_{j\bar{1}}^\dagger = -S_j^z f_j, \quad \bar{c}_{j\bar{1}} = -f_j^\dagger S_j^z, \quad (43)$$

where the string operators $K_j = \exp(i\pi \sum_{i<j} \bar{n}_i) = \prod_{i<j} \prod_{\alpha} (1 - 2\bar{n}_{i\alpha})$ are the natural generalizations of the ones introduced before. The number operators are defined as $\bar{n}_i = \bar{n}_{i1} + \bar{n}_{i\bar{1}}$ ($\bar{n}_{i\alpha} = \bar{c}_{i\alpha}^{\dagger} \bar{c}_{i\alpha}$). These f -operators have the remarkable property that

$$\{f_j^{\dagger}, f_j\} = \{S_j^+, S_j^-\}, \quad (44)$$

which suggests an analogy between spin operators and ‘‘constrained’’ fermions.

The generalization to two-flavor (or $s = 1/2$) HC anyons is achieved by the isomorphic map

$$\begin{cases} S_j^+ = \sqrt{2} (\bar{a}_{j1}^{\dagger} K_j(\theta) + K_j^{\dagger}(\theta) \bar{a}_{j\bar{1}}) \\ S_j^- = \sqrt{2} (K_j^{\dagger}(\theta) \bar{a}_{j1} + \bar{a}_{j\bar{1}}^{\dagger} K_j(\theta)) \\ S_j^z = \bar{n}_{j1} - \bar{n}_{j\bar{1}} \end{cases} \quad (45)$$

where the non-local transmutator ($\bar{n}_{j\alpha} = \bar{a}_{j\alpha}^{\dagger} \bar{a}_{j\alpha}$, $\bar{n}_j = \bar{n}_{j1} + \bar{n}_{j\bar{1}}$)

$$K_j(\theta) = \exp\left(i\theta \sum_{i<j} \bar{n}_i\right) = \prod_{i<j} \left(1 + (\exp[i\theta] - 1) \bar{n}_i\right) \quad (46)$$

($\bar{n}_{j\alpha} \bar{n}_{j\beta} = \delta_{\alpha\beta} \bar{n}_{j\alpha}$) allows rotation of the statistics of the particles whose algebra is determined by ($\mathbf{i} \leq \mathbf{j}$)

$$\begin{cases} [\bar{a}_{i\alpha}, \bar{a}_{j\beta}]_{\theta} = [\bar{a}_{i\alpha}^{\dagger}, \bar{a}_{j\beta}^{\dagger}]_{\theta} = 0, \\ [\bar{a}_{i\alpha}, \bar{a}_{j\beta}^{\dagger}]_{-\theta} = \delta_{ij} \begin{cases} 1 - e^{-i\theta} \bar{n}_{j\alpha} - \bar{n}_j & \text{if } \alpha = \beta, \\ -e^{-i\theta} \bar{a}_{j\beta}^{\dagger} \bar{a}_{j\alpha} & \text{if } \alpha \neq \beta, \end{cases} \end{cases} \quad (47)$$

In this case a more restrictive version of the Pauli exclusion principle applies where one can accommodate no more than a single particle per site regardless of α , i.e., $\bar{a}_{j\alpha}^{\dagger} \bar{a}_{j\beta}^{\dagger} = 0$, $\forall(\alpha, \beta)$.

4.3.1 Example: the one-dimensional fermionic Hubbard model

The Hubbard model [23] is the most popular model of a strongly interacting system in condensed matter physics. It contains a kinetic energy term represented by a hopping integral t plus a local on-site Coulomb repulsion U . The single-band Hubbard Hamiltonian is ($\sigma = \uparrow, \downarrow$)

$$H_{\text{Hubb}}^{1d} = t \sum_{j,\sigma} (c_{j\sigma}^{\dagger} c_{j+1\sigma} + c_{j+1\sigma}^{\dagger} c_{j\sigma}) + U \sum_{j=1}^{N_s} (\hat{n}_{j\uparrow} - \frac{1}{2}) (\hat{n}_{j\downarrow} - \frac{1}{2}), \quad (48)$$

Let us introduce a new transformation that is not a generalized Jordan-Wigner mapping but that works in the case of Hamiltonian H_{Hubb}^{1d} . The map is defined by

$$\begin{cases} S_{j1}^+ = c_{j\uparrow}^{\dagger} \bar{K}_{j\uparrow}, \\ S_{j1}^z = \hat{n}_{j\uparrow} - \frac{1}{2}, \\ S_{j2}^+ = c_{j\downarrow}^{\dagger} \bar{K}_{j\downarrow}, \\ S_{j2}^z = \hat{n}_{j\downarrow} - \frac{1}{2}, \end{cases} \quad (49)$$

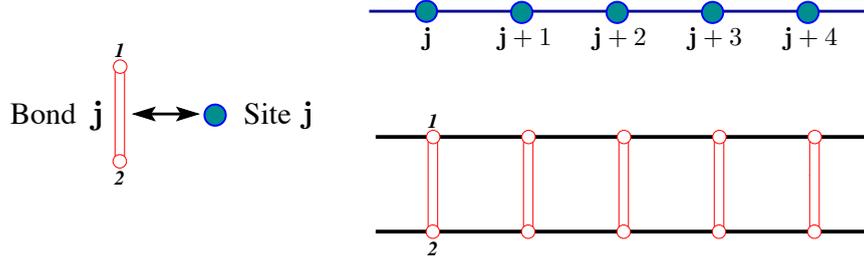


Fig. 6: Fermionic Hubbard chain maps onto a spin $S = 1/2$ ladder.

where the nonlocal operator $\bar{K}_{j\sigma}$ is defined as

$$\bar{K}_{j\uparrow} = \exp\left(i\pi\left(\sum_1 \hat{n}_{1\downarrow} + \sum_{1 < j} \hat{n}_{1\uparrow}\right)\right), \quad \bar{K}_{j\downarrow} = \exp\left(i\pi \sum_{1 < j} \hat{n}_{1\downarrow}\right). \quad (50)$$

Using this transformation, the one-dimensional Hubbard Hamiltonian becomes [9] ($\nu = 1, 2$)

$$H_{\text{Hubb}}^{1d} = 2t \sum_{j,\nu}^{N_s-1} (S_{j\nu}^x S_{j+1\nu}^x + S_{j\nu}^y S_{j+1\nu}^y) + U \sum_{j=1}^{N_s} S_{j1}^z S_{j2}^z, \quad (51)$$

which represents a two-leg ladder made out of two XY-chains coupled by an Ising interaction. The beauty of this map is that the fermionic Hubbard model and its dynamics can, in principle, be simulated with ultracold atoms in optical lattices or other quantum simulators.

4.3.2 Example: $su(N)$ spin-particle mappings

The fundamental (quark) representation of $su(N)$ can be mapped onto an algebra of constrained fermions ($\bar{c}_{j\alpha}^\dagger = \bar{a}_{j\alpha}^\dagger(\theta = \pi)$) or HC bosons ($\bar{b}_{j\alpha}^\dagger = \bar{a}_{j\alpha}^\dagger(\theta = 0)$) with $N_f = N-1$ flavors

$$\begin{aligned} \mathcal{S}^{\alpha\beta}(\mathbf{j}) &= \bar{a}_{j\alpha}^\dagger \bar{a}_{j\beta} - \delta_{\alpha\beta}/N \\ \mathcal{S}^{\alpha 0}(\mathbf{j}) &= \bar{a}_{j\alpha}^\dagger K_j^\theta, \quad \mathcal{S}^{0\beta}(\mathbf{j}) = (K_j^\theta)^\dagger \bar{a}_{j\beta} \\ \mathcal{S}^{00}(\mathbf{j}) &= \frac{N_f}{N} - \sum_{\alpha=1}^{N_f} \bar{n}_{j\alpha} = - \sum_{\alpha=1}^{N_f} \mathcal{S}^{\alpha\alpha}(\mathbf{j}), \end{aligned} \quad (52)$$

where $1 \leq \alpha, \beta \leq N_f$ runs over the set of particle flavors, and $\bar{a}_{j\alpha}^\dagger = \bar{a}_{j\alpha}^\dagger \prod_{\beta=1}^{N_f} (1 - \bar{n}_{j\beta})$. $\mathcal{S}^{\alpha_1\beta_1}(\mathbf{j})$ ($0 \leq \alpha_1, \beta_1 \leq N_f$) are the components of the $SU(N)$ -spin (i.e., there are N^2-1 linear independent components). It is easy to verify that these are generators of an $su(N)$ Lie algebra satisfying the commutation relations

$$[\mathcal{S}^{\alpha_1\alpha'_1}(\mathbf{j}), \mathcal{S}^{\beta_1\beta'_1}(\mathbf{j})] = \delta_{\alpha'_1\beta_1} \mathcal{S}^{\alpha_1\beta'_1}(\mathbf{j}) - \delta_{\alpha_1\beta'_1} \mathcal{S}^{\beta_1\alpha'_1}(\mathbf{j}). \quad (53)$$

For instance, for $N = 3$ we have ($\alpha = 1, 2$)

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_j & (K_j^\theta)^\dagger \bar{a}_{j1} & (K_j^\theta)^\dagger \bar{a}_{j2} \\ \bar{a}_{j1}^\dagger K_j^\theta & \bar{n}_{j1} - \frac{1}{3} & \bar{a}_{j1}^\dagger \bar{a}_{j2} \\ \bar{a}_{j2}^\dagger K_j^\theta & \bar{a}_{j2}^\dagger \bar{a}_{j1} & \bar{n}_{j2} - \frac{1}{3} \end{pmatrix}. \quad (54)$$

We can immediately see that the 2×2 block matrix $\mathcal{S}^{\alpha\beta}(\mathbf{j})$ ($1 \leq \alpha, \beta \leq 2$) contains the generators of $su(2)$. In general, from the commutation relations (53), we can verify that if $\mathcal{S}^{\alpha_1\beta_1}(\mathbf{j})$ are generators of $su(N)$, then $\mathcal{S}^{\alpha\beta}(\mathbf{j})$ are the generators of the subalgebra $su(N-1)$. This will be useful in Section 5.4.

5 Some applications

We have so far developed an algebraic framework for interacting quantum systems that will allow us to study complex phenomena characterized by the coexistence and competition of various broken symmetry states [3-5]. We have also proved a theorem that allowed us to connect all possible languages used in the quantum description of matter. Connecting the various languages through isomorphic mappings enables us to relate seemingly different physical phenomena, unveil hidden symmetries (i.e., uncover the *accidental degeneracies* of the original physical system), and, in some limiting cases, obtain the exact spectrum of the problem (or of a set of orthogonal subspaces). The ultimate goal is to use that framework to explore those unconventional complex states of matter from a unified perspective.

5.1 Generalized Jordan-Wigner transformations

In a similar fashion, one could continue for higher spin S irreps and would find that HC particles have $N_f = 2S$ flavors (we call these generalized Jordan-Wigner particles) [12]. Of course, this is not the only way to proceed. For example, for half-odd integer cases where $2S + 1 = 2^{\bar{N}_f}$ a simple transformation in terms of standard canonical multiflavor fermions is possible [12, 9].

We next generalize the JW spin-fermion mapping to any irreducible representation S [12]. Our mappings are valid for regular lattices in any spatial dimension d and particle statistics. These generalized JW mappings constitute a quantum version of the well-known classical spin-lattice-gas transformations. The significance of these transformations is that they help us understand various aspects of the same physical system by transforming intricate interaction terms in one representation into simpler ones in the other. Problems which seem intractable can even be exactly solved after the mapping. In other cases, new and better approximations can, in principle, be realized since fundamental symmetries which are hidden in one representation are manifest in the other. From a physical viewpoint, what our spin-particle transformations achieve is an exact connection between models of localized quantum spins S to models of itinerant particles with $(2S = N_f)$ color degrees of freedom or “effective” spin $s = S - \frac{1}{2}$.

We will consider now a type of fermions which naturally emerges from the strong coupling limit of models for interacting electrons. If the short range component of the Coulomb repulsion is much larger than the kinetic energy, the repulsion can be effectively replaced by a constraint of no double occupancy. This perturbative approach is usually implemented by a canonical transformation, which leads to an effective Hamiltonian acting on the subspace of states with no double occupancy. The fermionic subalgebra used to describe this effective model is generated by the so-called constrained fermions. Therefore, the constrained fermions are obtained by

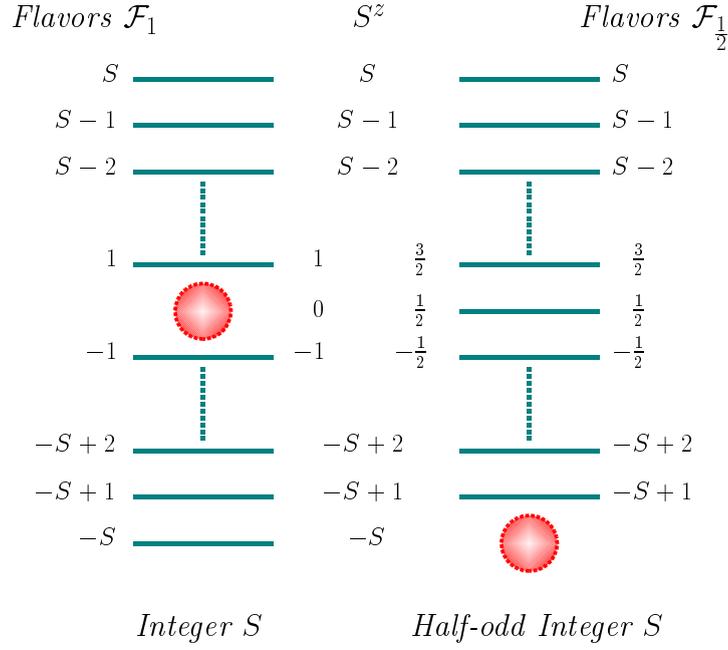


Fig. 7: Constrained fermion states per site for integer and half-odd integer spin S . There are $2S$ flavors and the corresponding $2S + 1$ values of S^z are shown in the middle column.

imposing to the canonical fermions a local constraint of no more than one particle per orbital (or site). This constraint may be incorporated into the fermionic algebra by defining the following creation and annihilation operators for the constrained fields

$$\bar{c}_{j\sigma}^\dagger = c_{j\sigma}^\dagger \prod_{\tau \in \mathcal{F}_\eta} (1 - \hat{n}_{j\tau}), \quad \bar{c}_{j\sigma} = \prod_{\tau \in \mathcal{F}_\eta} (1 - \hat{n}_{j\tau}) c_{j\sigma}, \quad (55)$$

where \mathcal{F}_η is the set of flavors, with $\eta = \frac{1}{2}$ or 1 depending upon the spin character of the irreducible representation. It is easy to check that the particles generated by this fermionic algebra satisfy the constraint of single occupancy, i.e., the eigenvalues of $\bar{n}_j = \sum_{\sigma=1}^{N_f} \bar{n}_{j\sigma}$ are either 0 or 1. The most well-known context where these fermions appear in condensed matter physics is the strong coupling limit of the Hubbard model, which leads to the t - J Hamiltonian [23]. The set of commutation relations those constrained fermions satisfy is

$$\begin{aligned} \{\bar{c}_{i\sigma}, \bar{c}_{j\sigma'}\} &= \{\bar{c}_{i\sigma}^\dagger, \bar{c}_{j\sigma'}^\dagger\} = 0, \\ \{\bar{c}_{i\sigma}, \bar{c}_{j\sigma'}^\dagger\} &= \delta_{ij} \begin{cases} 1 + \bar{n}_{j\sigma} - \bar{n}_j & \text{if } \sigma = \sigma', \\ \bar{c}_{j\sigma'}^\dagger \bar{c}_{j\sigma} & \text{if } \sigma \neq \sigma'. \end{cases} \end{aligned} \quad (56)$$

Notice that $\prod_{\tau \in \mathcal{F}_\eta, \tau \neq \sigma} (1 - \bar{n}_{j\tau}) = 1 + \bar{n}_{j\sigma} - \bar{n}_j$ with number operators satisfying $\bar{n}_{j\sigma} \bar{n}_{j\sigma'} = \delta_{\sigma\sigma'} \bar{n}_{j\sigma}$. The explicit form of the generalization is

Half-odd integer spin S : $\sigma \in \mathcal{F}_{\frac{1}{2}} = \{-S + 1, \dots, S\}$

$$\begin{aligned}
S_j^+ &= \eta_{\bar{S}} \bar{c}_{j\bar{S}+1}^\dagger K_j + \sum_{\substack{\sigma \in \mathcal{F}_{\frac{1}{2}} \\ \sigma \neq \bar{S}}} \eta_\sigma \bar{c}_{j\sigma+1}^\dagger \bar{c}_{j\sigma}, \\
S_j^- &= \eta_{\bar{S}} K_j^\dagger \bar{c}_{j\bar{S}+1} + \sum_{\substack{\sigma \in \mathcal{F}_{\frac{1}{2}} \\ \sigma \neq \bar{S}}} \eta_\sigma \bar{c}_{j\sigma}^\dagger \bar{c}_{j\sigma+1}, \\
S_j^z &= -S + \sum_{\sigma \in \mathcal{F}_{\frac{1}{2}}} (S + \sigma) \bar{n}_{j\sigma}, \\
\bar{c}_{j\sigma}^\dagger &= K_j^\dagger \prod_{\tau=-S}^{\sigma-1} \eta_\tau^{-1} (S_j^+)^{\sigma+S} \prod_{\tau \in \mathcal{F}_{\frac{1}{2}}} \frac{\tau - S_j^z}{\tau + S}.
\end{aligned} \tag{57}$$

Integer spin S : $\sigma \in \mathcal{F}_1 = \{-S, \dots, -1, 1, \dots, S\}$

$$\begin{aligned}
S_j^+ &= \eta_0 (\bar{c}_{j1}^\dagger K_j + K_j^\dagger \bar{c}_{j\bar{1}}) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}_{j\sigma+1}^\dagger \bar{c}_{j\sigma}, \\
S_j^- &= \eta_0 (K_j^\dagger \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger K_j) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}_{j\sigma}^\dagger \bar{c}_{j\sigma+1}, \\
S_j^z &= \sum_{\sigma \in \mathcal{F}_1} \sigma \bar{n}_{j\sigma}, \\
\bar{c}_{j\sigma}^\dagger &= K_j^\dagger \prod_{\tau=0}^{|\sigma|-1} \eta_\tau^{-1} \begin{cases} (S_j^+)^\sigma \prod_{\tau \in \mathcal{F}_1} \frac{\tau - S_j^z}{\tau} & \text{if } \sigma > 0, \\ (S_j^-)^\sigma \prod_{\tau \in \mathcal{F}_1} \frac{\tau - S_j^z}{\tau} & \text{if } \sigma < 0, \end{cases}
\end{aligned} \tag{58}$$

with $\eta_\sigma = \sqrt{(S - \sigma)(S + \sigma + 1)}$ (see Fig. 7). [A bar in a subindex means the negative of that number (e.g., $\bar{\sigma} = -\sigma$.)] These mappings enforce the condition on the Casimir operator $S_j^2 = S(S + 1)$. The generalized JW spin-fermion mapping can be easily extended to include a spin-anyon mapping simply by using the anyonic particles generated by $\bar{a}_{j\sigma}^\dagger$ and $\bar{a}_{j\sigma}$ [9].

5.2 Connecting seemingly unrelated phenomena: Haldane gap systems

Generically, half-odd integer spin chains have a qualitatively different excitation spectrum than integer spin chains. The Lieb, Schultz, Mattis, and Affleck theorem [24] establishes that the half-odd integer AF bilinear nearest-neighbors Heisenberg chain is gapless if the ground state (GS) is non-degenerate. The same model with integer spins is conjectured to have a Haldane gap [25]. To understand the origin of the Haldane gap we analyze the $S = 1$ XXZ Hamiltonian chain (an overall omitted constant $J > 0$ determines the energy scale)

$$H_{\text{xxz}} = \sum_j S_j^z S_{j+1}^z + \Delta (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) = \sum_j H_j^z + H_j^{\text{xx}}. \tag{59}$$

It is easy to show that the constrained fermion version of this Hamiltonian is a ($S = 1/2$) t - J_z model [26] plus particle non-conserving terms which break the $U(1)$ symmetry ($\sigma = \uparrow, \downarrow$)

$$H_{\text{xxz}} = \sum_{\mathbf{j}} (\bar{n}_{\mathbf{j}\uparrow} - \bar{n}_{\mathbf{j}\downarrow})(\bar{n}_{\mathbf{j}+1\uparrow} - \bar{n}_{\mathbf{j}+1\downarrow}) + \Delta \sum_{\mathbf{j}\sigma} \left(\bar{c}_{\mathbf{j}\sigma}^\dagger \bar{c}_{\mathbf{j}+1\sigma} + \bar{c}_{\mathbf{j}\sigma}^\dagger \bar{c}_{\mathbf{j}+1\bar{\sigma}} + \text{H.c.} \right). \quad (60)$$

The charge spectrum of the ($S = 1/2$) t - J_z model is gapless but the spin spectrum is gapped due to the explicitly broken $SU(2)$ symmetry (Luther-Emery liquid) [26]. Therefore, the spectrum of the $S = 1$ Hamiltonian associated with the t - J_z model, with $t = -\Delta$ and $J_z = 4$, (which has only spin excitations) is gapless. Hence the term which explicitly breaks $U(1)$ must be responsible for the opening of the Haldane gap. We can prove this by considering the perturbative effect that the pairing interaction $\eta \sum_{\mathbf{j}\sigma} (\bar{c}_{\mathbf{j}\sigma}^\dagger \bar{c}_{\mathbf{j}+1\bar{\sigma}} + \text{H.c.})$ has on the t - J_z Hamiltonian. To linear order in η (> 0), Eq. (60) maps onto the ($S = 1/2$) XYZ model with $\mathcal{J}_x = 2(\eta + \Delta)$, $\mathcal{J}_y = -2(\eta - \Delta)$, and $\mathcal{J}_z = -1$. To prove this statement we need to explain first how the lowest energy subspace of the t - J_z Hamiltonian can be mapped onto a spinless t - V model (the complete demonstration is presented in Ref. [26]).

The t - J_z Hamiltonian represents a hole-doped Ising model

$$H_{t-J_z} = \hat{H}_{J_z} + \hat{T} = J_z \sum_{\mathbf{j}} S_{\mathbf{j}}^z S_{\mathbf{j}+1}^z - t \sum_{\mathbf{j}\sigma} \left(\bar{c}_{\mathbf{j}\sigma}^\dagger \bar{c}_{\mathbf{j}+1\sigma} + \text{H.c.} \right). \quad (61)$$

Consider the set of parent states with M holes and $N_s - M = N_\uparrow + N_\downarrow$ quantum particles, $|\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle$, defined as

$$|\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \underbrace{|\uparrow\uparrow\downarrow\uparrow\downarrow\cdots\rangle}_{N_s-M} \underbrace{|\cdots\rangle}_M, \quad (62)$$

where $N_{\uparrow\uparrow}$ ($N_{\uparrow\downarrow}$) is the number of ferro (antiferro)-magnetic links ($N_{\uparrow\uparrow} + N_{\uparrow\downarrow} = N_s - M - 1$). These states are eigenstates of the magnetic part of H_{t-J_z} , \hat{H}_{J_z} , with energy $E_M(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) = J_z(N_{\uparrow\uparrow} - N_{\uparrow\downarrow})/4$, and z -component of the total spin $(N_\uparrow - N_\downarrow)/2$.

From a given parent state one can generate a subspace of the Hilbert space, $\mathcal{M}(N_\uparrow, N_{\uparrow\uparrow}, N_{\uparrow\downarrow})$, by applying the hopping operators $\hat{T}_{\mathbf{j},\sigma} = c_{\mathbf{j}\sigma}^\dagger c_{\mathbf{j}+1\sigma} + \text{H.c.}$ ($\mathbf{j} = 1, \dots, N_s - 1$) to the parent state and its descendants

$$|\Phi_1(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \hat{T}_{N_s-M,\sigma} |\Phi_0(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle \quad (63)$$

or, in general,

$$|\Phi_n(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle = \hat{T}_{\mathbf{j},\sigma} |\Phi_m(N_{\uparrow\uparrow}, N_{\uparrow\downarrow})\rangle. \quad (64)$$

The dimension \mathcal{D} of the subspace $\mathcal{M}(N_\uparrow, N_{\uparrow\uparrow}, N_{\uparrow\downarrow})$ is $\binom{N_s}{M}$. Moreover, these different subspaces are orthogonal and not mixed by the Hamiltonian H_{t-J_z} .

We want to show now that, for a given number of holes M , the subspace generated by the Néel parent state, $\mathcal{M}(N_\uparrow, 0, N_{\uparrow\downarrow}) \equiv \mathcal{M}_0$, contains the GS. To this end, one has to note that the matrix elements $\langle \Phi_n(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) | \hat{T} | \Phi_m(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) \rangle$ are the same for the different subspaces \mathcal{M} . Nonetheless, the magnetic matrix elements $\langle \Phi_n(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) | \hat{H}_{J_z} | \Phi_m(N_{\uparrow\uparrow}, N_{\uparrow\downarrow}) \rangle = \delta_{nm} A(\mathcal{N}_{\uparrow\uparrow}, \mathcal{N}_{\uparrow\downarrow})$ are

different for the different subspaces, with $A(0, \mathcal{N}_{\uparrow\downarrow}) \leq A(\bar{\mathcal{N}}_{\uparrow\uparrow}, \bar{\mathcal{N}}_{\uparrow\downarrow})$, $\mathcal{N}_{\uparrow\downarrow} = \bar{\mathcal{N}}_{\uparrow\uparrow} + \bar{\mathcal{N}}_{\uparrow\downarrow}$. [Notice that, for a generic state of a given subspace, $\mathcal{N}_{\uparrow\uparrow} + \mathcal{N}_{\uparrow\downarrow} \leq N_s - M - 1$ with the equality satisfied by the parent state only, where $\mathcal{N}_{\uparrow\uparrow} = N_{\uparrow\uparrow}$ and $\mathcal{N}_{\uparrow\downarrow} = N_{\uparrow\downarrow}$.] Therefore, the Hamiltonian matrices $H_{n,m}^{\mathcal{M}}$ (of dimension $\mathcal{D} \times \mathcal{D}$) in each subspace \mathcal{M} , consists of identical off-diagonal matrix elements ($H_{n,m}^{\mathcal{M}} = H_{n,m}^{\mathcal{M}'}$, $n \neq m$) and different diagonal ones. These Hermitian matrices can be ordered according to the increasing value of the energy E_M of their parent states, which is equivalent (for fixed N_s and M) to ordering by the increasing number of ferromagnetic (FM) links $N_{\uparrow\uparrow}$, $H^{\mathcal{M}} \equiv H^{N_{\uparrow\uparrow}}$. For any $N_{\uparrow\uparrow} < N'_{\uparrow\uparrow}$, $H^{N'_{\uparrow\uparrow}} = H^{N_{\uparrow\uparrow}} + B$, where B is a positive semidefinite matrix. Then, the monotonicity theorem tells us that

$$E_k(N_{\uparrow\uparrow}) \leq E_k(N'_{\uparrow\uparrow}) \quad \forall k = 1, \dots, \mathcal{D}, \quad (65)$$

where $E_k(N_{\uparrow\uparrow})$ are the eigenvalues of $H^{N_{\uparrow\uparrow}}$ arranged in increasing order. Therefore, we conclude that the lowest eigenvalue of H_{t-J_z} must be in \mathcal{M}_0 and is $E_1(0)$.

The next step consists in showing that, within the GS subspace \mathcal{M}_0 , the Hamiltonian H_{t-J_z} maps into an attractive spinless fermion model. If one makes the following identification

$$|\underbrace{\uparrow\downarrow\uparrow\downarrow\cdots}_{N_s-M} \underbrace{\circ\circ\circ\circ\cdots}_M\rangle \rightarrow |\underbrace{\bullet\bullet\bullet\cdots}_{N_s-M} \underbrace{\circ\circ\circ\circ\cdots}_M\rangle, \quad (66)$$

i.e., any spin particle ($\bar{c}_{j\sigma}^\dagger$, independently of the value of σ) maps into a single spinless fermion (d_j^\dagger) in \mathcal{M}_0 , it is straightforward to realize that all matrix elements of H^0 are identical to the matrix elements of

$$H_0 = -t \sum_j (d_j^\dagger d_{j+1} + \text{H.c.}) - \frac{J_z}{4} \sum_j \hat{n}_j \hat{n}_{j+1} \quad (67)$$

in the corresponding new basis, with $\hat{n}_j = d_j^\dagger d_j$.

The addition of the pairing term $\eta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}}^\dagger + \text{H.c.})$ to the $t-J_z$ Hamiltonian has two different effects in the lowest energy subspace \mathcal{M}_0 . The process where the pair of up and down particles created preserves the Néel ordering of the spins can be mapped into the creation of two spinless particles in the effective spinless model. The other possible process creates at least one FM link in the parent state and connects \mathcal{M}_0 with the subspace containing one FM link, $N_{\uparrow\uparrow} = 1$, (the lowest spin excitation) \mathcal{M}_1 . This means that the subspaces \mathcal{M} are no longer invariant under the application of the Hamiltonian. However this second process contributes to second order in η (η^2/Δ_s) due to the existence of a spin gap between the GSs of \mathcal{M}_0 and \mathcal{M}_1 . Therefore to first order in η , \mathcal{M}_0 is still an invariant subspace and the reduced Hamiltonian is a spinless model with a pairing (superconducting) term

$$H'_0 = H_0 + \eta \sum_j (d_j^\dagger d_{j+1}^\dagger + d_j d_{j+1}). \quad (68)$$

For arbitrary values of J_z , t , and hole density ν , H'_0 is equivalent (via the traditional Jordan-Wigner transformation) to the spin-1/2 XYZ chain Hamiltonian (up to an irrelevant constant)

$$H'_0 = \sum_j (\mathcal{J}_x s_j^x s_{j+1}^x + \mathcal{J}_y s_j^y s_{j+1}^y + \mathcal{J}_z (s_j^z s_{j+1}^z + s_j^z)), \quad (69)$$

and $\mathcal{J}_x = 2(\eta - t)$, $\mathcal{J}_y = -2(\eta + t)$, and $\mathcal{J}_z = -\frac{J_z}{4}$. In the language of our original Hamiltonian H_{xxz} , Eq. (60), $\mathcal{J}_x = 2(\eta + \Delta)$, $\mathcal{J}_y = -2(\eta - \Delta)$, and $\mathcal{J}_z = -1$. From exact solution of this model [27], it is seen that the system is critical only when $\eta = 0$ while for $\eta \neq 0$ a gap to all excitations opens.

It is important to note that the GS of the t - J_z model, $|\Psi_0^0\rangle$, has the same topological long-range order as the valence-bond-solid (Haldane state) [28, 26, 3], i.e., the correlation function [29] $\langle \Psi_0^0 | S_j^z \exp(i\pi \sum_{i=j+1}^{j+r-1} S_i^z) S_{j+r}^z | \Psi_0^0 \rangle = -\langle \Psi_0^0 | n_j n_{j+r} | \Psi_0^0 \rangle$ has a power law decay as a function of distance r to a constant value at the t - J_z point. This means that the superconducting term in H_{xxz} , although it is opening a gap (Haldane gap), does not change the topological order characterizing the GS. Therefore, the GS is in the same Haldane phase for $0 \leq \eta \leq \Delta$. In the particle language, the Haldane gap is a superconducting gap. Since each hole ($S^z = 0$ state in the spin language) is an anti-phase boundary (soliton) for the Néel ordering, the AF correlation function is short ranged for the GS of the t - J_z model. As demonstrated above, these solitonic excitations are massless at the t - J_z point, but become massive (gapped) as soon as the superconducting term is turned on ($\eta \neq 0$). As the superconducting term is derived from the transverse part of the Heisenberg interaction, it will not restore the AF ordering along the z direction. In this way it is easy to understand why the spin-spin correlation function, of the $S = 1$ AF Heisenberg chain is short ranged: $\langle S_j \cdot S_{j+r} \rangle \sim e^{-r/\xi}$ with ξ the correlation length.

5.3 Unveiling hidden symmetries and exactly-solvable points

We next illustrate the power of these transformations by showing exact solutions to lattice models previously unsolved by standard techniques [15]. The key is the existence of a general set of $SU(N)$ spin-particle transformations, fundamental to understanding the order hidden in complex behavior. More simply put, through the generalized spin-particle transformations, features that are subtle and hard to identify in one representation (hidden symmetries) can become prominent and easy to analyze in another (explicit symmetries). Indeed, these mappings *connect* seemingly unrelated physical phenomena, establishing equivalence relations among them. In the hierarchical group, all elements of the operator basis are symmetry generators. This allows one to study the coexistence and competition of phases, like ferromagnetism and BE condensation [15], with the corresponding OPs derived from the subgroup generators embedded in the largest global symmetry group of the problem.

Consider the $SU(2)$ -invariant model Hamiltonian

$$H_\phi = J\sqrt{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (\cos \phi \mathbf{S}_i \cdot \mathbf{S}_j + \sin \phi (\mathbf{S}_i \cdot \mathbf{S}_j)^2), \quad (70)$$

where $J > 0$ and summation is over bonds $\langle \mathbf{i}, \mathbf{j} \rangle$ of a regular d -dimensional hypercubic lattice with N_s sites and coordination z . A spin $S = 1$ operator \mathbf{S}_i is associated with lattice site \mathbf{i} and locally satisfies the $su(2)$ Lie algebra. The case $H_1 = H_{\phi=\frac{5\pi}{4}}$, which as we will see displays a global $SU(3)$ symmetry, can be conveniently written in a ($s = \frac{1}{2}$) HC boson representation

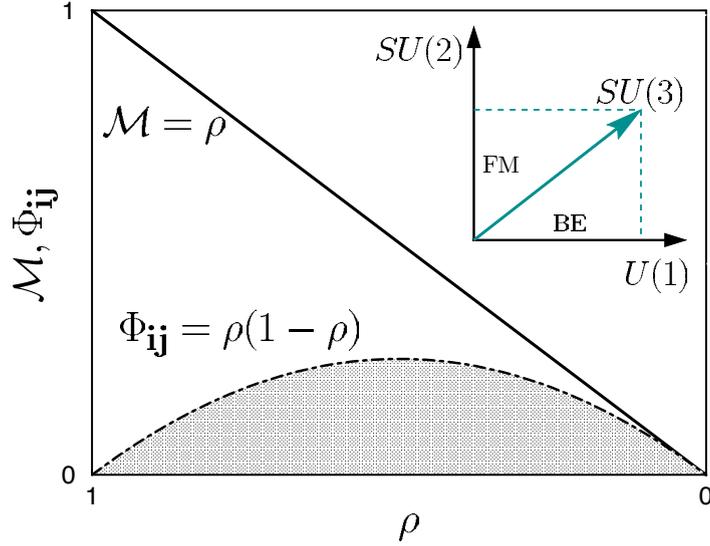


Fig. 8: Coexistence of FM and BE condensation. The correlation function $\Phi_{\uparrow\uparrow}(\mathbf{i}, \mathbf{j}) = \Phi_{\mathbf{i}, \mathbf{j}}$ is expressed in the thermodynamic limit. The inset schematically displays the $SU(3)$ OP living in an eight-dimensional space with projections onto the $SU(2)$ (FM) and $U(1)$ (BE) axes.

($\bar{n}_{j\sigma} \bar{n}_{j\sigma'} = \delta_{\sigma\sigma'} \bar{n}_{j\sigma}$), Eq. (14), as an extended t - J like Hamiltonian,

$$H_1 = -J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle, \sigma} \left(\bar{b}_{\mathbf{i}\sigma}^\dagger \bar{b}_{\mathbf{j}\sigma} + \text{H.c.} \right) - 2J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \mathbf{s}_i \cdot \mathbf{s}_j - 2J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(1 - \frac{\bar{n}_i + \bar{n}_j}{2} + \frac{3}{4} \bar{n}_i \bar{n}_j \right), \quad (71)$$

with $\mathbf{s}_j = \frac{1}{2} \bar{b}_{j\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} \bar{b}_{j\beta}$ an $s = \frac{1}{2}$ operator ($\boldsymbol{\sigma}$ denoting Pauli matrices), and $\bar{n}_j = \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\uparrow} + \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\downarrow}$. This last form in turn can be rewritten as

$$H_1 = -2J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} P_s(\mathbf{i}, \mathbf{j}), \quad (72)$$

where $P_s(\mathbf{i}, \mathbf{j}) = P_s^2(\mathbf{i}, \mathbf{j})$ is the projector onto the symmetric subspace ($S = 0, 2$) corresponding to the bond $\langle \mathbf{i}, \mathbf{j} \rangle$ which indicates that if one finds a state that is symmetric under the permutation of nearest neighbors \mathbf{r}_i and \mathbf{r}_j , then that state is the GS.

For a system of \mathcal{N} HC bosons the GS is

$$|\Psi_0(\mathcal{N}, S_z)\rangle = (\hat{b}_{0\uparrow}^\dagger)^{\mathcal{N}_\uparrow} (\hat{b}_{0\downarrow}^\dagger)^{\mathcal{N}_\downarrow} |0\rangle, \quad (73)$$

($\mathcal{N} = \mathcal{N}_\uparrow + \mathcal{N}_\downarrow \leq N_s$) with an energy $E_0 = -JN_s z$ and a total $S_z = \frac{\mathcal{N}_\uparrow - \mathcal{N}_\downarrow}{2}$. The operator $\hat{b}_{0\sigma}^\dagger$ is the $\mathbf{k} = \mathbf{0}$ component of $\bar{b}_{j\sigma}^\dagger$, i.e., $\hat{b}_{\mathbf{k}\sigma}^\dagger = \frac{1}{\sqrt{N_s}} \sum_{\mathbf{j}} e^{i\mathbf{k}\cdot\mathbf{r}_j} \bar{b}_{j\sigma}^\dagger$. The quasihole and quasiparticle excited states are

$$\begin{cases} |\Psi_{\mathbf{k}}^h(\mathcal{N}, S_z)\rangle = \hat{b}_{\mathbf{k}\sigma} |\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasihole,} \\ |\Psi_{\mathbf{k}}^p(\mathcal{N}, S_z)\rangle = \hat{b}_{\mathbf{k}\sigma}^\dagger |\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasiparticle,} \end{cases} \quad (74)$$

with the excitation energy of each being $\omega_{\mathbf{k}} = Jz(\frac{1}{z} \sum_{\nu} e^{i\mathbf{k}\cdot\mathbf{e}_\nu} - 1)$ where the sum runs over the vectors \mathbf{e}_ν which connect a given site to its z nearest neighbors. In the $|\mathbf{k}| \rightarrow 0$ limit, $\omega_{\mathbf{k}} \rightarrow 0$.

Clearly the GS in Eq. (73) is a FM BE condensate with any partial spin polarization, and the form of the result is *independent of the spatial dimensionality of the lattice*. We note that different values of S_z correspond to the different orientations of the magnetization \mathcal{M} associated to the broken $SU(2)$ spin rotational symmetry of the GS. We also note that the degeneracy of states with different number of particles \mathcal{N} indicates a broken $U(1)$ charge symmetry (conservation of the number of particles) associated to the BE condensate. A signature of BE condensation is the existence of off-diagonal long-range order (ODLRO) in the correlation function $\Phi_{\sigma\sigma'}(\mathbf{ij}) = \langle \bar{b}_{i\sigma}^\dagger \bar{b}_{j\sigma'} \rangle$. When \mathcal{N}_\uparrow and \mathcal{N}_\downarrow are both of order N_s , there are two eigenvectors with eigenvalues of order N_s and the condensate is thus a mixture.

We can easily compute the magnetization \mathcal{M} and phase coherence of these various (non-normalized) degenerate GSs for a given density $\rho = \frac{\mathcal{N}}{N_s}$. For example, in the fully polarized case, $\mathcal{N} = \mathcal{N}_\uparrow$, $\mathcal{M} = \langle S_j^z \rangle = \rho$, and the ODLRO $\Phi_{\uparrow\uparrow}(\mathbf{ij}) = \frac{\rho(1-\rho)}{1-\epsilon}$ ($\mathbf{r}_i \neq \mathbf{r}_j$), where $\epsilon = 1/N_s$. Similarly, the two-particle correlation function $\langle \Delta_i^\dagger \Delta_j \rangle = \Phi_{\uparrow\uparrow}(\mathbf{ij}) \frac{(\rho-\epsilon)(1-\rho-\epsilon)}{(1-2\epsilon)(1-3\epsilon)}$, where $\Delta_i^\dagger = \bar{b}_{i\uparrow}^\dagger \bar{b}_{i+\delta\uparrow}^\dagger$. Therefore the exact GS has two spontaneously broken continuous symmetries (see Fig. 8).

The exact solution defines the features of the phase diagram that our proposed framework must qualitatively admit. We will see below that both OPs (magnetization and phase), as promised, are embedded in an $SU(3)$ order parameter. We remark that the phase coexistence in the boson representation maps back to a $S = 1$ FM phase coexisting with another spin phase. To see this consider the state $|\Psi_0(\mathcal{N}, 0)\rangle$ for which $\langle S_i^z \rangle = \langle S_i^x \rangle = \langle S_i^y \rangle = 0$, which implies that it is a singlet state in the $S = 1$ representation. We will show below that this other phase has a pure spin-nematic ordering.

5.4 Identifying order parameters

The theory of phase transitions starts with Landau's pioneering work in 1937 [13,14]. One of his achievements was the realization of the fundamental relation between spontaneous symmetry breaking and the OP that measures this violation, thus giving simple prescriptions to describe order in terms of irreducible representations of the symmetry group involved. Another was the development of a phenomenological scheme to study the behavior of systems near a phase transition. Landau's theory has been successfully applied to study phase transitions where thermal fluctuations are most relevant. The field of quantum phase transitions studies the changes that can occur in the macroscopic properties of matter at zero temperature due to changes in the parameters characterizing the system. While one generally knows what to do if the OP is known, Landau's postulate gives no procedure for finding it. In this section we describe a simple algebraic framework for identifying OPs.

We have seen that the local OP acquires its simplest form when it is expressed in terms of the HL. In addition, the generators of this language exhaust all possible local OPs which may result from the solution of the problem under consideration. In other words, any local OP can be written as a linear combination of generators of the HL. Consider the bilinear-biquadratic $S = 1$ model of Eq. (70). The parameter ϕ sets the relative strength between the bilinear and bi-

quadratic terms. As shown below, there are four isolated values of $\phi = \{\pi/4, \pi/2, 5\pi/4, 3\pi/2\}$ for which H_ϕ is $SU(3)$ invariant. The only symmetry which is present for any value of ϕ is the global $SU(2)$ invariance since H_ϕ is a function of the scalar products $\mathbf{S}_i \cdot \mathbf{S}_j$.

The Hamiltonian H_ϕ has been the subject of several studies in the last two decades [30–35], nevertheless, the complete characterization of the different phases is not completely solved. A semiclassical treatment for $d > 1$ [30] indicates that there are four different phases: the usual FM ($\pi/2 < \phi < 5\pi/4$) and AF ($3\pi/2 < \phi < \pi/4$) phases are separated on both sides by collinear- ($5\pi/4 < \phi < 3\pi/2$) and orthogonal-nematic ($\pi/4 < \phi < \pi/2$) orderings. We will show below that the collinear- and orthogonal-nematic phases obtained with the semiclassical approximation are replaced by uniform- and staggered-nematic orderings, respectively.

As we have seen in previous section, the $SU(3)$ spins in the fundamental representation and the $S = 1$ $SU(2)$ spins are two equivalent languages. In addition, we have shown in Sections 4.3.2 and 4.3 that the $SU(3)$ spins and the $S = 1$ $SU(2)$ spins can be respectively mapped onto $s = 1/2$ HC bosons. We will use now these transformations to map the spin one Hamiltonian H_ϕ onto its $SU(3)$ spin version. For pedagogical reasons, it is convenient to use the $s = 1/2$ HC bosons as an intermediate language.

In Section 4.3.2, we introduced a spin-particle transformation connecting $SU(N)$ spins and multiflavored HC bosons (JW particles, in general). In particular, the fundamental (*quark*) representations of $su(N)$ were mapped onto an algebra of HC bosons with $N_f = N - 1$ flavors (see Eq. (52)). For $N = 3$ the HC bosons have two flavors ($\alpha = \uparrow, \downarrow$) which can be associated to an internal spin $s = 1/2$ degree of freedom. A compact way of writing the $SU(3)$ spin in terms of HC bosons is

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_j & \bar{b}_{j\uparrow} & \bar{b}_{j\downarrow} \\ \bar{b}_{j\uparrow}^\dagger & \bar{n}_{j\uparrow} - \frac{1}{3} & \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\downarrow} \\ \bar{b}_{j\downarrow}^\dagger & \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\uparrow} & \bar{n}_{j\downarrow} - \frac{1}{3} \end{pmatrix}. \quad (75)$$

In the same way we wrote in Eq. (75) the generators of $SU(3)$ in the fundamental representation, we can write down the corresponding expressions for the generators in the conjugate representation

$$\tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_j & -\bar{b}_{j\downarrow}^\dagger & -\bar{b}_{j\uparrow}^\dagger \\ -\bar{b}_{j\downarrow} & \bar{n}_{j\downarrow} - \frac{1}{3} & \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\downarrow} \\ -\bar{b}_{j\uparrow} & \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\uparrow} & \bar{n}_{j\uparrow} - \frac{1}{3} \end{pmatrix}. \quad (76)$$

When the $S=1$ operators are replaced by the corresponding functions of $SU(3)$ generators in the fundamental and the conjugate representations, it turns out that H_ϕ , up to an irrelevant constant, is a linear combination of the FM and the AF $SU(3)$ Heisenberg models ($0 \leq \alpha_1, \beta_1 \leq 2$)

$$H_\phi = J\sqrt{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\cos \phi \mathcal{S}^{\alpha_1 \beta_1}(\mathbf{i}) \mathcal{S}^{\beta_1 \alpha_1}(\mathbf{j}) + (\sin \phi - \cos \phi) \mathcal{S}^{\alpha_1 \beta_1}(\mathbf{i}) \tilde{\mathcal{S}}^{\beta_1 \alpha_1}(\mathbf{j}) \right). \quad (77)$$

Repeated Greek superindices are summed and the site index \mathbf{i} runs over one of the two sublattices. This expression for H_ϕ illustrates the very important result that any nonlinear interaction in the original representation is simply a bilinear term in the new representation when mapped

onto the highest rank algebra [15]. In particular, as mentioned above, there are certain special points in parameter space where the Hamiltonian is highly symmetric. For example, for $\phi = \frac{\pi}{4}$ and $\frac{5\pi}{4}$, H_ϕ is explicitly invariant under uniform $SU(3)$ transformations on the spins [36], while for $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$, H_ϕ is explicitly invariant under staggered conjugate rotations of the two sublattices. These symmetries are hard to identify in the original spin representation but are manifest in the $SU(3)$ representation.

In the following we will concentrate on the determination of the quantum phase diagram of H_ϕ for spatial dimensions $d > 1$ to avoid the strong effects of quantum fluctuations which can restore the continuous symmetry when $d = 1$. In the previous section we have analyzed the high symmetry point $\phi = \frac{5\pi}{4}$. We found that the GS has a non-zero OP

$$\mathcal{S} = \sum_{\mathbf{j}} \mathcal{S}(\mathbf{j}) , \quad (78)$$

associated to a broken continuous $SU(3)$ symmetry. This order parameter is the uniform $SU(3)$ magnetization and corresponds to the coexistence of a FM and a uniform spin-nematic ordering (see Table 1). This indicates that $\phi = \frac{5\pi}{4}$ is a quantum phase transition point separating a FM phase from a uniform spin-nematic one. Let us consider now the related point $\phi = \frac{\pi}{4}$ which differs in an overall sign from the previous case. This sign changes the interaction from FM to AF. Therefore, for this new high symmetry point we expect to get a GS characterized by the staggered order parameter

$$\mathcal{S}_{\text{ST}} = \sum_{\mathbf{j}} \exp(i\mathbf{Q} \cdot \mathbf{j}) \mathcal{S}(\mathbf{j}) , \quad (79)$$

where \mathbf{Q} is the AF wave vector. It is clear from the left column of Table 1, that this staggered $SU(3)$ OP corresponds to the coexistence of the staggered $SU(2)$ magnetization

$$\mathbf{M}_{\text{ST}} = \sum_{\mathbf{j}} \exp(i\mathbf{Q} \cdot \mathbf{j}) \mathbf{S}_{\mathbf{j}} , \quad (80)$$

and the staggered nematic OP

$$\mathbf{N}_{\text{ST}} = \sum_{\mathbf{j}} \exp(i\mathbf{Q} \cdot \mathbf{j}) \mathbf{N}_{\mathbf{j}} . \quad (81)$$

$\mathbf{N}_{\mathbf{j}}$ is the symmetric and traceless component of the tensor obtained from the tensorial product of two vectors $\mathbf{S}_{\mathbf{j}}$. Hence, $\phi = \frac{\pi}{4}$ is a transition point separating the usual AF ordering from a staggered spin-nematic phase characterized by the OP of Eq. (81).

We will consider now the other two high-symmetry points, $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$. For $\phi = \frac{3\pi}{2}$, the $SU(3)$ symmetry is generated by the staggered operator

$$\mathcal{S}_+ = \sum_{\mathbf{j} \in A} \mathcal{S}(\mathbf{j}) + \sum_{\mathbf{j} \in B} \tilde{\mathcal{S}}(\mathbf{j}) , \quad (82)$$

where A and B denote the two different sublattices of a hypercubic lattice. In this case, we have a FM interaction between $\mathcal{S}(\mathbf{i})$ and $\tilde{\mathcal{S}}(\mathbf{j})$, and then \mathcal{S}_+ is the OP characterizing the broken

Table 1: Generators of OPs and its relations for three different languages $\mathcal{A} \wedge \Gamma_A$. Each column represents a language, in this case $\dim \Gamma_A = D = 3$. *M* stands for magnetism, *SN* spin-nematic, *BE* Bose-Einstein condensation, and *CDW* charge-density wave. $su(3) \wedge FR$ is the hierarchical language with *FR* meaning fundamental representation.

	$su(2) \wedge S = 1$	HC bosons $\wedge \alpha = 2$	$su(3) \wedge FR$	
M	$\begin{cases} S^x = \frac{1}{\sqrt{2}}(\mathcal{S}^{01} + \mathcal{S}^{20} + \mathcal{S}^{02} + \mathcal{S}^{10}) \\ S^y = \frac{-1}{\sqrt{2}i}(\mathcal{S}^{01} + \mathcal{S}^{20} - \mathcal{S}^{02} - \mathcal{S}^{10}) \\ S^z = \mathcal{S}^{11} - \mathcal{S}^{22} \end{cases}$	M	$\begin{cases} s^x = \frac{1}{2}(\mathcal{S}^{12} + \mathcal{S}^{21}) \\ s^y = \frac{1}{2i}(\mathcal{S}^{12} - \mathcal{S}^{21}) \\ s^z = \frac{1}{2}(\mathcal{S}^{11} - \mathcal{S}^{22}) \end{cases}$	$\mathcal{S}^{\alpha\beta}$
SN	$\begin{cases} (\mathcal{S}^x)^2 = \frac{2}{3} + \frac{1}{2}(\mathcal{S}^{12} + \mathcal{S}^{21} + \mathcal{S}^{00}) \\ (\mathcal{S}^z)^2 = \frac{2}{3} - \mathcal{S}^{00} \\ \{\mathcal{S}^x, \mathcal{S}^y\} = i(\mathcal{S}^{21} - \mathcal{S}^{12}) \\ \{\mathcal{S}^x, \mathcal{S}^z\} = \frac{1}{\sqrt{2}}(\mathcal{S}^{01} - \mathcal{S}^{20} - \mathcal{S}^{02} + \mathcal{S}^{10}) \\ \{\mathcal{S}^y, \mathcal{S}^z\} = \frac{-1}{\sqrt{2}i}(\mathcal{S}^{01} - \mathcal{S}^{20} + \mathcal{S}^{02} - \mathcal{S}^{10}) \end{cases}$	BE	$\begin{cases} \bar{b}_\uparrow^\dagger = \mathcal{S}^{10} \\ \bar{b}_\downarrow^\dagger = \mathcal{S}^{20} \\ \bar{b}_\uparrow = \mathcal{S}^{01} \\ \bar{b}_\downarrow = \mathcal{S}^{02} \end{cases}$	$\alpha, \beta \in [0, 2]$
		CDW	$\bar{n} = \frac{2}{3} - \mathcal{S}^{00}$	$\text{Tr } \mathcal{S} = 0$

$SU(3)$ symmetry of the GS. It is interesting to note that when the $SU(3)$ OP \mathcal{S}_+ is reduced with respect to the $SU(2)$ group, the two coexisting OPs are the staggered magnetization (see Eq. (80)) and the *uniform* nematic OP

$$\mathbf{N} = \sum_{\mathbf{j}} \mathbf{N}_{\mathbf{j}}. \quad (83)$$

In other words, if we apply an $SU(3)$ rotation generated by \mathcal{S}_+ to the staggered magnetization we get the uniform nematic order parameter, and vice versa, the uniform nematic OP is rotated into the staggered magnetization. This can be immediately seen by writing down the components of the local $SU(2)$ magnetization and the nematic OP as a function of the local generators of $su(3)$ in the conjugate representation $\tilde{\mathcal{S}}(\mathbf{j})$

$$\begin{aligned} S^x &= \frac{-1}{\sqrt{2}}(\tilde{\mathcal{S}}^{01} + \tilde{\mathcal{S}}^{20} + \tilde{\mathcal{S}}^{02} + \tilde{\mathcal{S}}^{10}), \\ S^y &= \frac{1}{\sqrt{2}i}(\tilde{\mathcal{S}}^{01} + \tilde{\mathcal{S}}^{20} - \tilde{\mathcal{S}}^{02} - \tilde{\mathcal{S}}^{10}), \\ S^z &= \tilde{\mathcal{S}}^{22} - \tilde{\mathcal{S}}^{11}, \\ (\mathcal{S}^x)^2 - \frac{2}{3} &= \frac{1}{2}(\tilde{\mathcal{S}}^{12} + \tilde{\mathcal{S}}^{21} + \tilde{\mathcal{S}}^{00}), \quad (\mathcal{S}^z)^2 - \frac{2}{3} = -\tilde{\mathcal{S}}^{00}, \\ \{\mathcal{S}^x, \mathcal{S}^y\} &= i(\tilde{\mathcal{S}}^{21} - \tilde{\mathcal{S}}^{12}), \\ \{\mathcal{S}^x, \mathcal{S}^z\} &= \frac{1}{\sqrt{2}}(\tilde{\mathcal{S}}^{01} - \tilde{\mathcal{S}}^{20} - \tilde{\mathcal{S}}^{02} + \tilde{\mathcal{S}}^{10}), \\ \{\mathcal{S}^y, \mathcal{S}^z\} &= \frac{-1}{\sqrt{2}i}(\tilde{\mathcal{S}}^{01} - \tilde{\mathcal{S}}^{20} + \tilde{\mathcal{S}}^{02} - \tilde{\mathcal{S}}^{10}). \end{aligned} \quad (84)$$

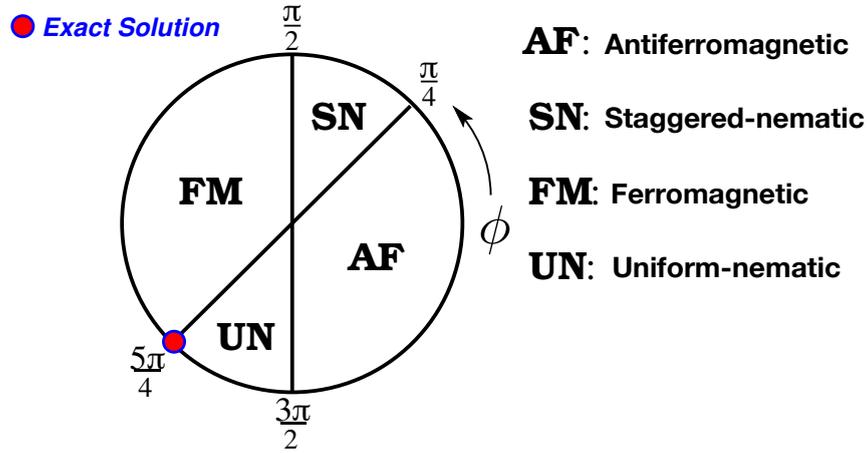


Fig. 9: Quantum phase diagram of the bilinear-biquadratic $S = 1$ model for $d \geq 2$.

Comparing these expressions to the ones in Table 1, we see that when we change from $\mathcal{S}(\mathbf{j})$ to $\tilde{\mathcal{S}}(\mathbf{j})$, there is a change in sign for the three components associated to the magnetization, while the five components corresponding to the nematic parameter remain the same. Then, it is clear that \mathcal{S}_+ describes the coexistence of a staggered magnetization and a uniform nematic ordering. Therefore, the conclusion is that $\phi = \frac{3\pi}{2}$ separates an ordinary AF phase from the uniform nematic ordering.

The last high symmetry point to be considered is $\phi = \frac{\pi}{2}$. In this case the coupling between $\mathcal{S}(\mathbf{i})$ and $\tilde{\mathcal{S}}(\mathbf{j})$ turns out to be positive, i.e., AF, and therefore we expect to get a broken continuous symmetry characterized by the OP

$$\mathcal{S}_- = \sum_{\mathbf{j} \in A} \mathcal{S}(\mathbf{j}) - \sum_{\mathbf{j} \in B} \tilde{\mathcal{S}}(\mathbf{j}) . \quad (85)$$

From the considerations above, it is clear that \mathcal{S}_- describes the coexistence of ferromagnetism (uniform magnetization) and staggered nematic order. Hence, $\phi = \frac{\pi}{2}$ is a transition point separating these two phases.

ϕ	Global $SU(3)$ OP	OP 1	OP 2
$5\pi/4$ (FM-UN)	$\mathcal{S} = \sum_{\mathbf{j}} \mathcal{S}(\mathbf{j})$	M	N
$\pi/4$ (AF-SN)	$\mathcal{S}_{ST} = \sum_{\mathbf{j}} e^{i\mathbf{Q} \cdot \mathbf{j}} \mathcal{S}(\mathbf{j})$	\mathbf{M}_{ST}	\mathbf{N}_{ST}
$3\pi/2$ (AF-UN)	$\mathcal{S}_+ = \sum_{\mathbf{j} \in A} \mathcal{S}(\mathbf{j}) + \sum_{\mathbf{j} \in B} \tilde{\mathcal{S}}(\mathbf{j})$	\mathbf{M}_{ST}	N
$\pi/2$ (FM-SN)	$\mathcal{S}_- = \sum_{\mathbf{j} \in A} \mathcal{S}(\mathbf{j}) - \sum_{\mathbf{j} \in B} \tilde{\mathcal{S}}(\mathbf{j})$	M	\mathbf{N}_{ST}

Table 2: Order parameters describing the different phases of the bilinear-biquadratic $S = 1$ Heisenberg model for $d > 1$. ϕ indicates the phase boundary where the two phases in parentheses coexist.

In this way, by identifying the high-symmetry points of H_ϕ we have determined the quantum phase diagram of this model (see Fig. 9). In addition to the transition points, we have obtained explicit expressions for the OPs associated to each phase for any $d > 1$; they are summarized in Table 2. We can also predict from this analysis that the four transition points (high-symmetry points) correspond to first-order quantum phase transitions. In each phase, the corresponding OP has a finite value and they coexist pairwise at the high-symmetry points. However, as soon as we depart from this point in one or the other direction in ϕ , the $SU(3)$ symmetry is removed and one of the OPs goes discontinuously to zero. In other words, the states with pure magnetic (FM or AF) and nematic orderings belong to different representations of $SU(2)$ (the remaining symmetry) so only one of them remains as the GS when the $SU(3)$ symmetry is lifted.

In closing this section let us summarize the main steps to follow in order to obtain and classify the local OPs.

- Identify the group $\mathcal{G}_{\text{HL}} = SU(D)$ associated to the HL whose fundamental representation has the same dimension D as the local Hilbert space of the problem. The generators of this language exhaust all possible local OPs.
- Identify the group of global symmetries of the Hamiltonian \mathcal{G} which are direct products of local transformations.
- Given that $\mathcal{G} \subseteq \mathcal{G}_{\text{HL}}$, one can classify the generators of \mathcal{G}_{HL} in the fundamental representation according to the irreps of \mathcal{G} . Each irrep leads to a different broken symmetry OP.
- Key: existence of a general set of $SU(D)$ transformations.

5.5 Hierarchical mean-field theories

In this section we are interested in using our algebraic framework to develop new approximation schemes whenever exact solutions are not available. In previous sections we outlined a framework to identify OPs based upon isomorphic mappings to a HL defined by the set of operators which in the fundamental representation (of dimension D) has the largest number of symmetry generators of the group. Any local operator can be expressed as a *linear* combination of the generators of the HL. The building of the HL depends upon the dimension D of the local Hilbert space, \mathcal{H}_j , modeling the physical phenomena. For instance, if one is modeling a doped AF insulator with a t - J Hamiltonian [23], then $D = 3$ (i.e., there are three possible states per site) and a HL is generated by a basis of the Lie algebra $su(3)$ in the fundamental representation [15]. As explained and proved in Refs. [15, 9], there is always a HL associated to each physical problem. These ideas complement Landau's concept of an OP providing a mechanism to reveal them, something that is outside the groundwork of his theory. Indeed, Landau's theory does not say what the OPs should be in a general situation.

As mentioned above, these isomorphic mappings not only unveil hidden symmetries of the original physical system but also manifestly establish equivalences between seemingly unrelated physical phenomena. Nonetheless, this is not sufficient to determine the *exact* phase diagram of

the problem: One has to resort to either numerical simulations with their well-known limitations or, as will be shown in here, to a *guided* approximation which at least preserves the qualitative nature of the possible thermodynamic states. A key observation in this regard is the fact that typical model Hamiltonian operators written in the HL become quadratic in the symmetry generators of the hierarchical group, and this result is independent of the group of symmetries of the Hamiltonian. This suggests a simple approximation, based upon group theoretical grounds, which deals with competing orders on an equal footing and will be termed *hierarchical mean-field theory* (HMFT) [37, 38]. In a sense, that will become clear below, HMFT constitutes the *optimum* MF or saddle-point solution that approximates the energy and correlation functions of the original problem. The HMFT is distinctly suitable when the various phases displayed by a system are the result of competing interactions and non-linear couplings of their constituents matter fields.

Since the $su(N)$ languages provide a complete set of HLs [9], any model Hamiltonian can be written in a similar fashion once we identify the appropriate HL and apply the corresponding SW mapping in the *fundamental representation* (the ordering operators will, of course, have a different meaning and algebraic expressions). The key point is that the Hamiltonian operator in the HL becomes quadratic in the symmetry generators of the hierarchical group ($SU(3)$ in the present case).

The idea behind any MF approximation is to disentangle interaction terms into quadratic ones replacing some of the elementary mode operators by their mean value. The crux of our HMFT is that the approximation is done in the HL where all possible local OPs are treated on an equal footing and the number of operators replaced by their mean value is minimized since the Hamiltonian is quadratic in the symmetry generators. In this way, the information required is minimal. In mathematical terms, given $\mathcal{O}_{ij}^\dagger \mathcal{O}_{ij} = \langle \mathcal{O}_{ij}^\dagger \rangle \mathcal{O}_{ij} + \mathcal{O}_{ij}^\dagger \langle \mathcal{O}_{ij} \rangle - \langle \mathcal{O}_{ij}^\dagger \rangle \langle \mathcal{O}_{ij} \rangle + (\mathcal{O}_{ij}^\dagger - \langle \mathcal{O}_{ij}^\dagger \rangle)(\mathcal{O}_{ij} - \langle \mathcal{O}_{ij} \rangle)$, for an arbitrary bond-operator \mathcal{O}_{ij} , the approximation amounts to neglecting the latter fluctuations, i.e., $\mathcal{O}_{ij}^\dagger \mathcal{O}_{ij} \approx \langle \mathcal{O}_{ij}^\dagger \rangle \mathcal{O}_{ij} + \mathcal{O}_{ij}^\dagger \langle \mathcal{O}_{ij} \rangle - \langle \mathcal{O}_{ij}^\dagger \rangle \langle \mathcal{O}_{ij} \rangle$. An important result is that all local OPs are treated equally and, moreover, symmetries of the original Hamiltonian related to the OPs are not broken explicitly in certain limits. In a sense, this is the *best* MF approximation that can be performed, i.e., the best *non-interacting* Hamiltonian that approximates the energy and correlation functions of the original problem.

We study now a simple model which displays coexistence and competition between antiferromagnetism and BE condensation (superfluidity). The model represents a gas of interacting spin-1/2 HC bosons with Hamiltonian ($t > 0$)

$$H = t \sum_{\langle i,j \rangle, \sigma} \left(\bar{b}_{i\sigma}^\dagger \bar{b}_{j\sigma} + \text{H.c.} \right) + J \sum_{\langle i,j \rangle} (\mathbf{s}_i \cdot \mathbf{s}_j - \frac{\bar{n}_i \bar{n}_j}{4}) + V \sum_{\langle i,j \rangle} \bar{n}_i \bar{n}_j - \bar{\mu} \sum_{\mathbf{j}} \bar{n}_{\mathbf{j}}, \quad (86)$$

where $\mathbf{s}_{\mathbf{j}} = \frac{1}{2} \bar{b}_{\mathbf{j}\alpha}^\dagger \vec{\sigma}_{\alpha\beta} \bar{b}_{\mathbf{j}\beta}$ is an $s = \frac{1}{2}$ operator ($\vec{\sigma}$ denoting Pauli matrices). Notice that H is an extended t - J -like model of HC bosons instead of constrained fermions. These HC bosons could represent three-state atoms, like the ones used in trapped BE condensates (BECs), in an optical lattice potential. For the sake of simplicity we will only consider the AF, $J > 0$, case.

In the HL, H represents a Heisenberg-like Hamiltonian [12] in the presence of an external magnetic field μ' ($J_{\alpha_1\beta_1} = J_{\beta_1\alpha_1}$ with $0 \leq \alpha_1, \beta_1 \leq 2$)

$$H = \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\alpha_1\beta_1} \mathcal{S}^{\alpha_1\beta_1}(\mathbf{i}) \mathcal{S}^{\beta_1\alpha_1}(\mathbf{j}) - \mu' \sum_{\mathbf{j}} \mathcal{S}^{00}(\mathbf{j}), \quad (87)$$

with $J_{00} = V - J/2$, $J_{01} = J_{02} = t$, $J_{11} = J_{12} = J_{22} = J/2$, and $\mu' = \frac{z}{3}(2V - J/2) - \bar{\mu}$. This HL furnishes the natural framework to analyze the symmetries of the Hamiltonian H . There is always an $SU(2)$ spin symmetry generated by $\mathcal{S}^{11} - \mathcal{S}^{22}$, \mathcal{S}^{12} , and \mathcal{S}^{21} . When $\mu' = 0$ and $V = 2t$, there are five additional generators of symmetries related to the charge degrees of freedom. Moreover, if $J = V = 2t$ there is full $SU(3)$ symmetry. For $\mu' \neq 0$, the only charge symmetry that remains is a $U(1)$ symmetry generated by \mathcal{S}^{00} (conservation of the total charge). In this way the HL, leading to a unique OP from which all possible embedded orderings are derived, provides a unified description of the possible thermodynamic states of the system. Yet, it remains to establish the orderings that survive as a result of tuning the parameters of the Hamiltonian or external variables such as temperature and particle filling.

For arbitrary values of the parameters J/t and V/t we do not know a priori how to determine exactly the phase diagram of H (we know that for $J = V = 2t = \frac{2\bar{\mu}}{z} < 0$, we can find the exact GS and lowest energy states [15]). The resulting Hamiltonian ($V = 2t$ with no loss of generality) is up to irrelevant constant terms

$$H = - \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\frac{J}{2} A_{\mathbf{ij}}^\dagger A_{\mathbf{ij}} + t \sum_{\sigma=\uparrow, \downarrow} B_{\sigma\mathbf{ij}}^\dagger B_{\sigma\mathbf{ij}} \right) - \mu \sum_{\mathbf{j}} n_{\mathbf{j}0}, \quad (88)$$

where $\mu = zt - \bar{\mu}$ and the ordering operators

$$\begin{cases} A_{\mathbf{ij}}^\dagger = b_{\mathbf{i}\uparrow}^\dagger b_{\mathbf{j}\downarrow}^\dagger - b_{\mathbf{i}\downarrow}^\dagger b_{\mathbf{j}\uparrow}^\dagger \\ B_{\sigma\mathbf{ij}}^\dagger = b_{\mathbf{i}\sigma}^\dagger b_{\mathbf{j}0}^\dagger - b_{\mathbf{i}0}^\dagger b_{\mathbf{j}\sigma}^\dagger \end{cases} \quad (89)$$

which transform as singlets with respect to the generators of $SU(2)$ spin and charge symmetries, respectively: $[A_{\mathbf{ij}}^\dagger, \mathcal{S}^{12(21)}(\mathbf{i}) + \mathcal{S}^{12(21)}(\mathbf{j})] = 0 = [B_{\uparrow(\downarrow)\mathbf{ij}}^\dagger, \mathcal{S}^{10(20)}(\mathbf{i}) + \mathcal{S}^{10(20)}(\mathbf{j})]$.

The resulting MF Hamiltonian reads

$$\tilde{H} = - \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\frac{JA}{2} (A_{\mathbf{ij}}^\dagger + A_{\mathbf{ij}}) + tB \sum_{\sigma=\uparrow, \downarrow} (B_{\sigma\mathbf{ij}}^\dagger + B_{\sigma\mathbf{ij}}) \right) - \mu \sum_{\mathbf{j}} n_{\mathbf{j}0} + \lambda \sum_{\mathbf{j}, \alpha} n_{\mathbf{j}\alpha} \quad (90)$$

$$= \sum_{\mathbf{k} \in \text{RBZ}} \left(\Lambda_A b_{\mathbf{k}\uparrow}^\dagger b_{-\mathbf{k}+\mathbf{Q}\downarrow}^\dagger + \Lambda_B \sum_{\sigma=\uparrow, \downarrow} b_{\mathbf{k}\sigma}^\dagger b_{-\mathbf{k}+\mathbf{Q}0}^\dagger + \text{H.c.} + (\lambda - \mu) n_{\mathbf{k}0} + \lambda \sum_{\sigma=\uparrow, \downarrow} n_{\mathbf{k}\sigma} \right), \quad (91)$$

where the sum of momenta \mathbf{k} is performed over the reduced Brillouin zone (RBZ) with AF ordering wave vector \mathbf{Q} , with $\Lambda_A = -2JA\gamma_{\mathbf{k}}$, $\Lambda_B = -4tB\gamma_{\mathbf{k}}$, with $\gamma_{\mathbf{k}} = \frac{1}{z} \sum_{\vec{\delta}} e^{i\mathbf{k} \cdot \vec{\delta}}$ ($\vec{\delta}$ are nearest-neighbor vectors). Note that when $B = 0$, the $SU(2)$ spin and $U(1)$, \mathcal{S}^{00} , symmetries are conserved; the opposite case $A = 0$ preserves $\mathcal{S}^{10(01)} + \mathcal{S}^{20(02)}$ and $\mathcal{S}^{11} + \mathcal{S}^{22} - \mathcal{S}^{00}$ symmetries. In Eq. (91) we have only considered homogeneous solutions.

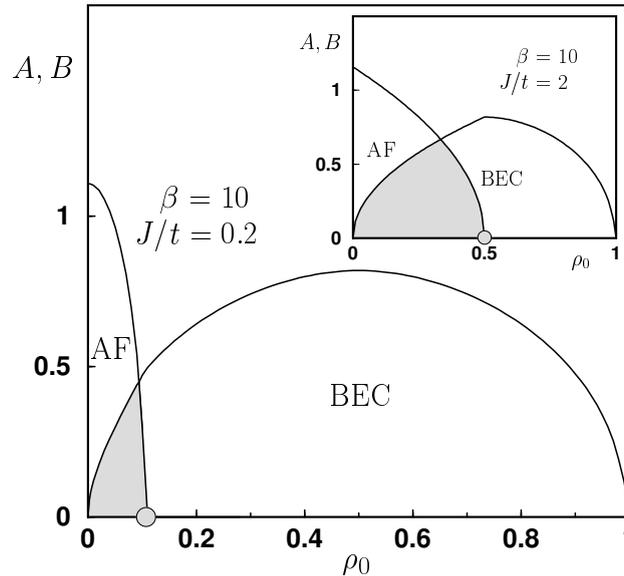


Fig. 10: Order fields A and B as a function of the density ρ_0 for different values of J/t and inverse temperature $\beta = 10$ (in units of t^{-1}). The filled circle on the density axis indicates a quantum critical point.

Generalization of these ideas to include clusters of many elementary degrees of freedom, e.g., quantum spins, can be found in [38]. In practice, we tile the original many-body lattice system into clusters preserving most of the symmetries of the Hamiltonian and represent each many-body state by the action of a composite operator over the vacuum of a new enlarged Fock space. The mapping that relates the original set of operators and the new composite ones is canonical if a physical constraint is implemented. As a consequence, the Hamiltonian of study can be exactly re-expressed in the new language of composite operators and treated by standard many-body techniques, with the advantage that the intra-cluster quantum correlations are computed exactly while the inter-cluster are dealt with in a MF way. The HMFT approach has been implemented successfully in a variety of frustrated strongly correlated system [39–43].

5.6 Quantum simulations

A new challenge in information theory and computer science has recently emerged as the result of exploiting the fundamental laws of quantum mechanics. This new set of ideas comprise what is known as “Theory of Quantum Computation and Quantum Information” and has as a major objective to process information in a way that exceeds the capabilities of classical information [44]. The device that performs the manipulation of information is named quantum computer and the standard unit of information is the *qubit* (i.e., a two-level system). The close relationship between information processing and the physical phenomena leading to it is perhaps the most remarkable aspect of this new paradigm. Since information can be represented in many different physical forms, and easily converted from one form to another without changing its meaning, quantum information represents a new abstract archetype for information processing independent of the precise implementation of the quantum computer, only requiring at least one physical representation to be useful.

A key fundamental concept in information theory is the realization [45, 46] that a model of computation is intimately connected to a physical system through a closed operator algebra. In other words, each physical system is associated to a certain language (e.g., spin $S = 1/2$) and thus to an algebra realizing it (e.g., Pauli algebra), and that particular algebra may become a possible model of computation. An immediate consequence is that an arbitrary physical system can be simulated by another physical system (e.g., a quantum computer) whenever there exists an isomorphic mapping between the different operator algebras representing the systems [45]. Simple examples are provided in Refs. [45], [46], and [47], where it is shown how to simulate a $1d$ impurity Anderson model using a quantum computer based on an array of spins $S=1/2$ [48]. It is very clear the power that our fundamental theorem has by providing the formal connections (isomorphisms) between the different languages of nature. Therefore, the implications for quantum information and computation are rather obvious, namely that one can identify quantum resources and define convenient models of computation, or imitate an arbitrary quantum phenomena with a given quantum computer given the appropriate dictionaries to translate nature's language to the *machine language*. In this way, one can recognize the subject of quantum simulations as one of those areas where the concepts of language and dictionaries developed in the present lecture are of particular relevance.

Physical phenomena can be simulated or *imitated* by a quantum network [45, 47, 49] with the help of a quantum computer. Imitation is realized through a quantum algorithm which consists of a quantum network with a means to repeat blocks of instructions. A quantum network is defined by a sequence of universal gates (unitary operations), applied to the system for the purpose of information processing, and measurements in a fixed temporal order. The measurement operation is mostly needed to classically access information about the state of the system. Every matrix which represents a reversible operation on quantum states can be expressed as a product of the one and two-qubit gates, and the minimum set needed to represent any such matrices is called a universal set of gates.

When trying to simulate a problem using quantum information processing, an important issue is to determine how many physical resources are needed for the solution. The main resources are *quantum space*, the number of qubits needed, and *quantum time*, the number of quantum gates required. The accounting of algorithmic resources forms the foundations of quantum complexity theory. One of the objectives in quantum information theory is to accomplish imitation efficiently, i.e., with polynomial complexity, and the hope is that quantum imitation is more efficient (i.e., needs less resources) than classical imitation. There are examples that support such hope (e.g., fermion simulations with polynomially bounded statistical errors [45, 47]), although there is no general proof that indicates the superiority of quantum over classical imitations, regarding efficiency. Indeed, there is, so far, no efficient quantum algorithm that can determine the GS (or, in general, the spectrum) of a given Hermitian operator [47], despite occasional claims. It is known that the ability to resolve this question leads to efficient algorithms for NP-hard problems like the traveling salesman conundrum.

A very important observation, in connection with the notion of efficiency, is a corollary of our fundamental theorem: Given two languages, the generators of one of them can be written as a

polynomial function, with *polynomial complexity in the number of modes or resources*, of the generators of the other and vice versa. This result implies that the important algorithmic step of *translation* from the language of the system to be imitated to the *machine language* does not change the complexity of the quantum space and time.

Certainly, a general purpose quantum computer is not the only device that allows simulation of physical phenomena in nature (with its many languages). Imitation can also be achieved in a conceptually different manner using a quantum simulator. The main distinction is the lack of universality of the latter. An example of a quantum simulator is an optical lattice [50] which is specifically designed to imitate a given physical Hamiltonian and where there is limited quantum control. The possibility of control and tunability of the interactions of the elementary constituents offers the potential to design new states of matter. This is of particular relevance in strongly correlated matter where these quantum simulators furnish the benchmark to test theories and approximations. Again, the importance of the languages and dictionaries developed in this manuscript is clear and concrete.

6 Concluding remarks

The development of exact algebraic methods is one of the most elegant and promising tools towards the complete understanding of quantum phases of matter and their corresponding phase transitions. We presented an algebraic framework aimed at uncovering the order behind the potential multiplicity of complex phases in interacting quantum systems, a paradigm at the frontiers of condensed matter physics. We argued that symmetry, and topology (not so much explained in this lecture notes), are key guiding principles behind such complex emergent behavior. Emphasis has been made in developing a systematic mathematical structure that allows one to attack these problems within a single unifying approach.

A key result, from which all other results follow, is the proof of a fundamental theorem that permits to connect the various operator languages used in the description of the properties of physical systems. This theorem together with the notion of transmutation of statistics provides the tools necessary to unify the quantum description of matter. To formalize this unification we needed to rigorously define the concepts of language and dictionary (isomorphism): To model a particular physical phenomena we commonly identify the main degrees of freedom of the problem and associate to them certain operators. One can furnish the resulting set of operators (that we call language) with an algebraic structure and ask whether two different languages have something in common. The fundamental theorem tells us that two languages can be connected whenever the dimension of their local Hilbert spaces are equal. We expanded the notion of local Hilbert space to embrace different Hilbert space decompositions (we saw, for instance, how to map the Hilbert space of a bond to a site). The resulting one-to-one language mappings we named dictionaries (a traditional example of which is the Jordan-Wigner mapping).

In the course of the presentation we showed, through example, many different dictionaries relating diverse operator languages. In this way we defined universality of behavior as an equivalence relation between seemingly different physical phenomena which share exactly the same

underlying mathematical structure as a result of one-to-one language mappings (for example, the spin nematic order and Bose-Einstein condensation of flavored hard-core bosons). If it is the whole system Hamiltonian that maps onto another in a different language (like the example we described above), the universality applies to all length and time scales. However, sometimes only particular invariant subspaces of the original Hamiltonian map onto another system Hamiltonian. In this case, universality is only manifested at certain energy scales. The t - J_z chain model provides a beautiful example of the latter situation: the low-energy manifold of states maps onto an XXZ model Hamiltonian, which can be exactly solved using the Bethe ansatz [26] (the t - J_z model is quasi-exactly solvable). Out of the many languages one can use to describe a given physical problem there is a class, we named hierarchical language, which has the advantage that any local operator can be expressed as a linear combination of its generators. In this way, hierarchical languages provide the tools necessary to classify order parameters.

There are several reasons why our algebraic framework constitutes a powerful method to unveil complex phenomena in interacting quantum systems. Most importantly: To connect seemingly unrelated physical phenomena (e.g., models for high-temperature superconductors or heavy-fermion systems and quantum spin theories); to identify general symmetry principles behind complex phase diagrams; to unveil hidden symmetries (and associated order parameters) to explore new states of matter with internal orders not contemplated before; to obtain exact solutions of relevant physical models that display complex ordering at certain points in Hamiltonian space; and to find new approximations which do not privilege any of the competing interactions. For instance, in the *hierarchical mean-field theory* approach, we approximated the dynamics (and thermodynamics) treating all possible local order parameters on an equal footing. One may say that this procedure follows the guiding principles of *maximum symmetry* and *minimum information*. This allowed us to obtain in a simple manner the phase diagram of a problem exhibiting coexistence and competition between antiferromagnetism and superfluidity. Combined with an analysis of fluctuations (to analyze the stability of the mean-field) one now has a simple machinery to design phase diagrams.

Several important concepts have been left out of this lecture notes. For instance, the notion of *emergent symmetry* [9], i.e., the fact that new symmetries not realized in the Hamiltonian describing the system can emerge at low energies [51]. There is one concept, in particular, that deserves special attention. This is the notion of a *duality transformation*. Dualities appear in nearly all disciplines of physics and play a central role in statistical mechanics and field theory [52, 53]. When available, these mathematical transformations provide an elegant and efficient way to obtain information about models that need not be exactly solvable. Most notably, dualities may be used to determine features of phase diagrams such as boundaries between phases, and the exact location of some critical/multicritical points. Historically, dualities were introduced in classical statistical mechanics by Kramers and Wannier [54] as a relation between the partition function of one system at high temperature (or weak coupling) to the partition function of another (dual) system at low temperatures (or strong coupling). This relation allowed for a determination of the exact critical temperature of the two-dimensional Ising model on a square lattice before the exact solution of the model was available. Later on, it was noticed that,

due to the connection between quantum theories in d space dimensions and classical statistical systems in $d + 1$ dimensions, dualities can provide relations between quantum theories in the strong coupling and weak coupling regimes [52]. Our work is motivated by a quest to realize a simple unifying framework for the detection and treatment of dualities.

An algebraic approach to dualities and self-dualities for systems of *arbitrary spatial dimensionality* d has been developed in Refs. [55, 10]. This theory of dualities is based on the notion of bond algebras [56, 10]. It deals with classical and quantum dualities in a unified fashion explaining the precise connection between quantum dualities and the low temperature (strong-coupling)/high temperature (weak-coupling) dualities of classical statistical mechanics (or (Euclidean) path integrals) [57]. Its range of applications includes discrete lattice, continuum field, and gauge theories. Dualities are revealed to be local, structure-preserving mappings between model-specific bond algebras that can be implemented as unitary transformations, or partial isometries if gauge symmetries are involved. This characterization permits to search systematically for dualities and self-dualities in quantum models of arbitrary system size, dimensionality and complexity, and any classical model admitting a transfer matrix or operator representation. In particular, special dualities like exact dimensional reduction, emergent, and gauge-reducing dualities that solve gauge constraints can be easily understood in terms of mappings of bond algebras. The transformations are, in general, quite non-local in the mapped degrees of freedom. Non-local transformations like dual variables and Jordan-Wigner dictionaries are algorithmically derived from the local mappings of bond algebras. This permits to establish a precise connection between quantum dual and classical disorder variables. Our bond-algebraic approach goes beyond the standard (Fourier transformation) approach to classical dualities (see, e.g., Appendix 1 of Ref. [10]), and could help resolve the long-standing problem of non-Abelian duality transformations [58]. Several interesting examples and applications, including location of phase boundaries, spectral behavior and, notably, how bond-algebraic dualities help constrain and realize fermionization in an arbitrary number of spatial dimensions, can be found in Refs. [10] and [59].

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- [20] A monoid is a triple $(\mathfrak{M}, \square, \mathbf{1})$ in which \mathfrak{M} is a non-empty set, \square is an associative product in \mathfrak{M} , and $\mathbf{1}$ is an element of \mathfrak{M} such that $\square(\mathbf{1}, \mathfrak{a}) = \mathfrak{a} = \square(\mathfrak{a}, \mathbf{1})$ for all $\mathfrak{a} \in \mathfrak{M}$. In this way we see that the concept of monoid generalizes the notion of group; a group is a monoid all of whose elements are invertible. A monoid can also be defined as a semigroup (\mathfrak{M}, \square) with an element that is the unit for \square , i.e., $\mathbf{1}$.
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