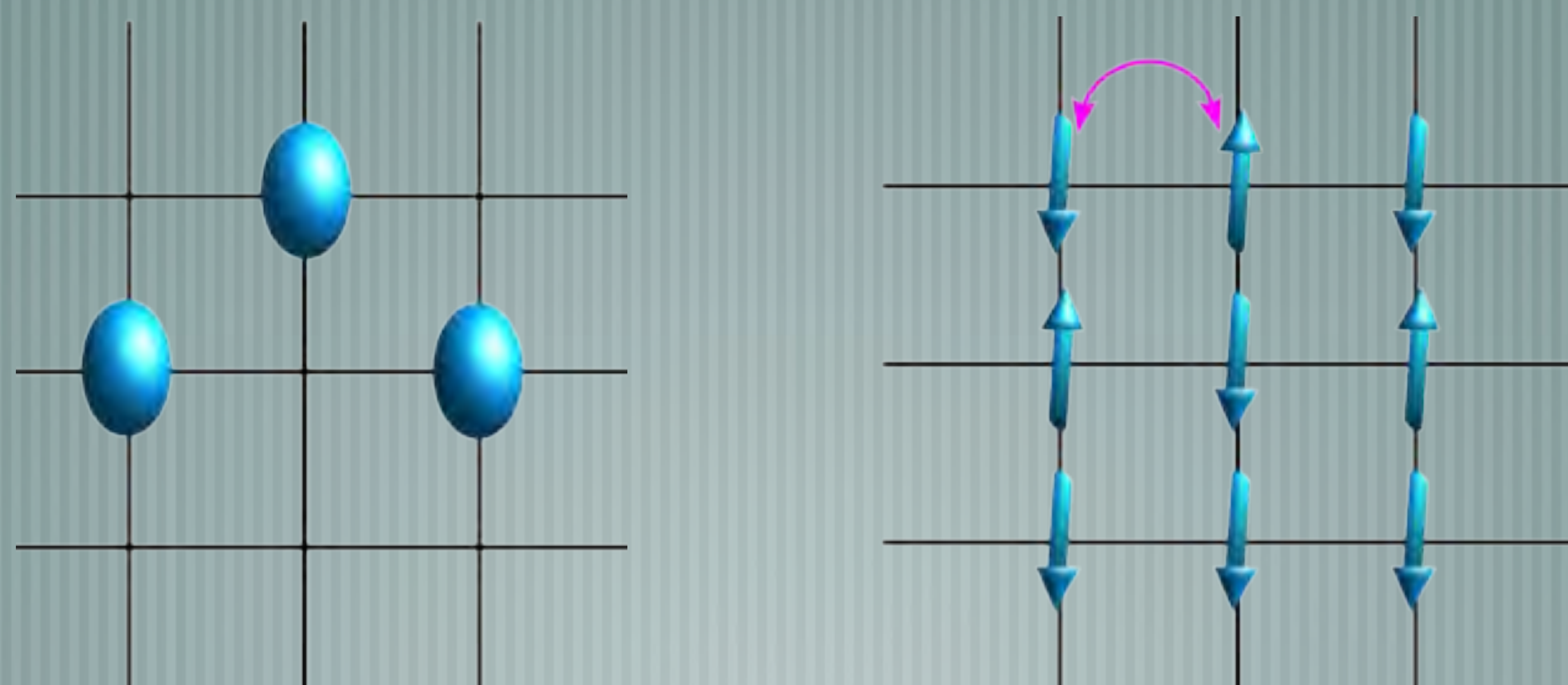
Algebraic Methods in Many-Body Physics

Gerardo Ortiz
Department of Physics - Indiana University

Particle Spin



Many-body Methods for Real Materials - September 2019

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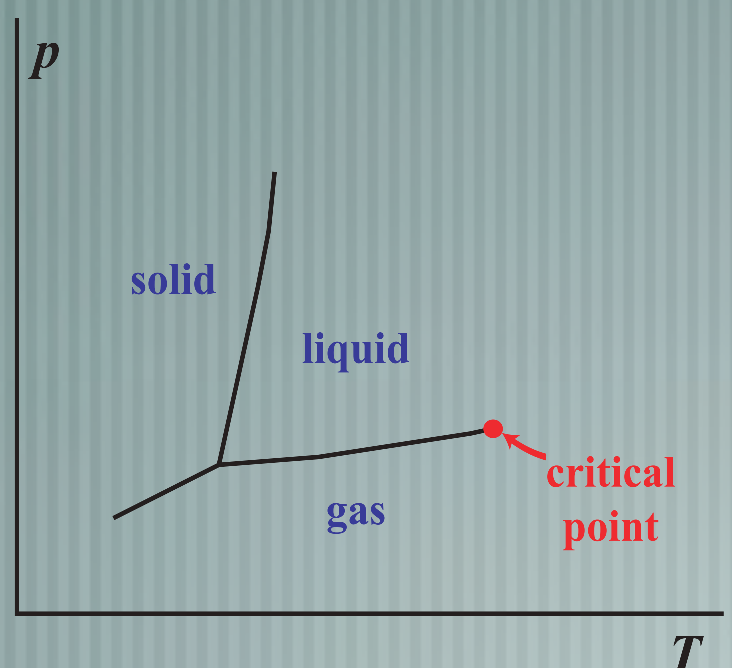
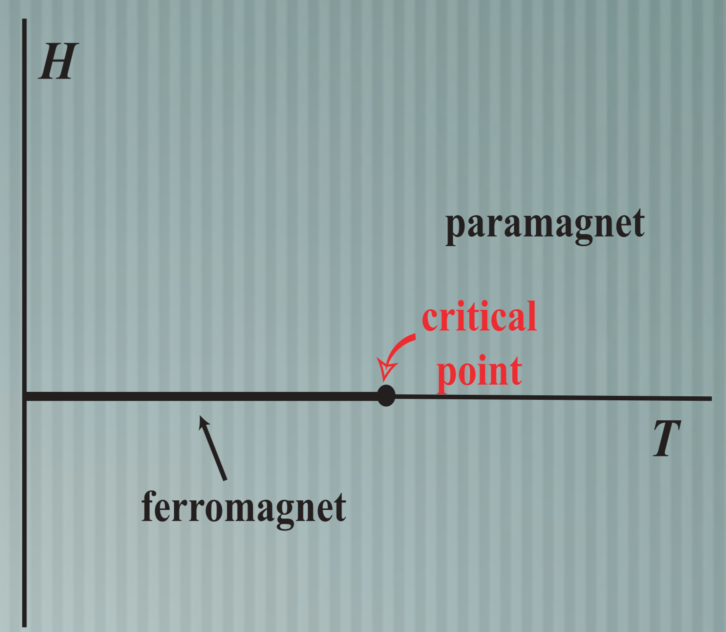
Liquid-Gas transition



Hidden Z_2 symmetry at the critical point

Ferromagnetic transition



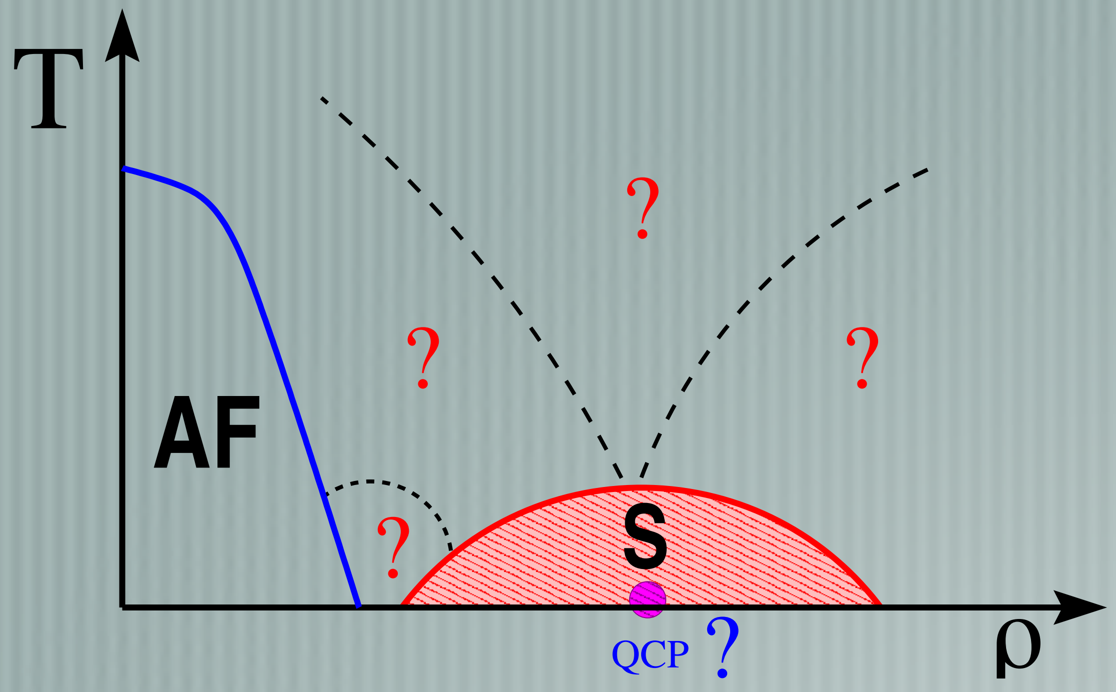
Z_2 symmetry

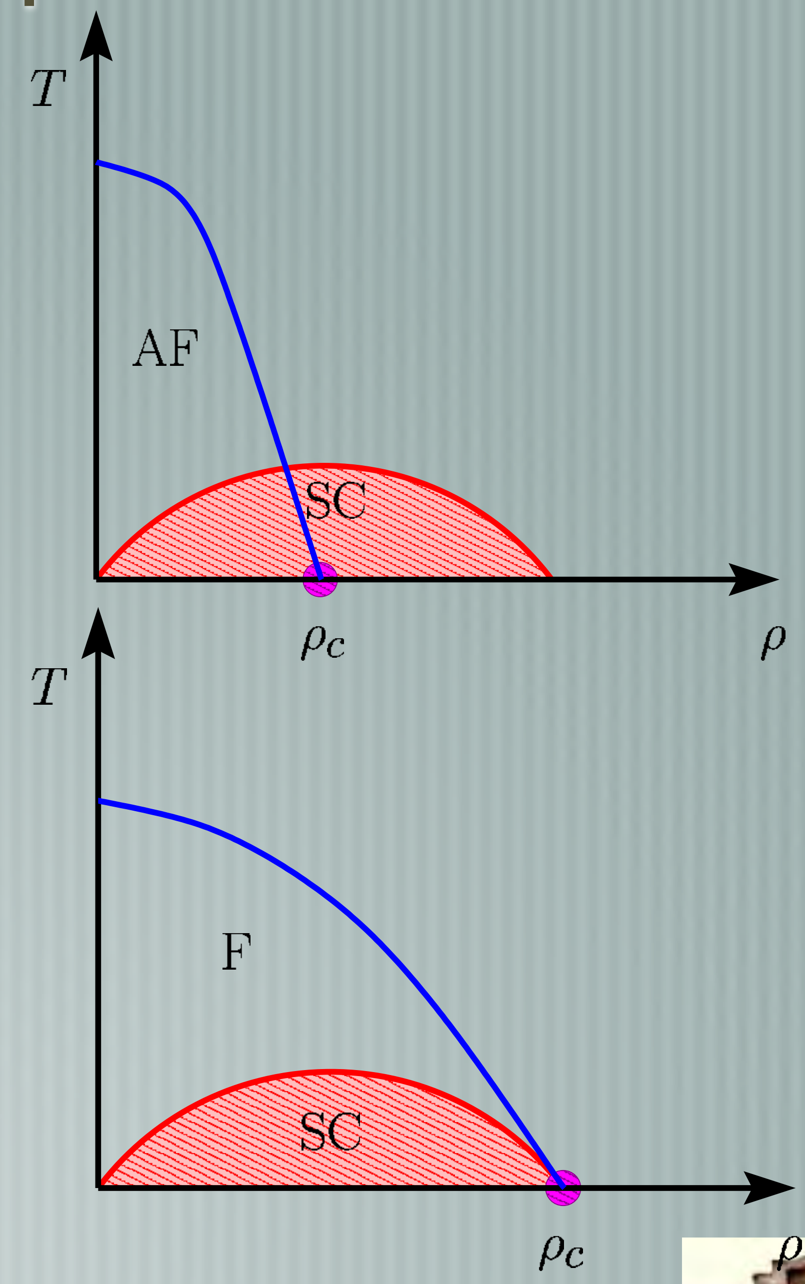
Motivation

- WHAT?
  - Unifying framework to study interacting quantum systems:
    Dictionaries connecting the different languages of quantum mechanics
    Connection between languages and symmetries
- WHY?
  - New paradigm: coexistence and competition of “non-trivial” phases
  - Determination and classification of order parameters
  - Identify general symmetry principles for complex phase diagrams
  - Unveil hidden symmetries to explore new states of matter
  - Connect seemingly unrelated physical phenomena
  - Obtain exact solutions and develop better approximation schemes
  - Duality Maps: Strong Coupling to Weak Coupling relations
  - Simulation of one system by another physical system

Coexistence and Competition of Complex Phases of Matter

- Emergence of a new paradigm in Physics of matter, characterized by coexistence and competition of "non-trivial" phases.





– Absence of a small parameter:
  Standard mean-field theories do not work

– Lack of exact solutions

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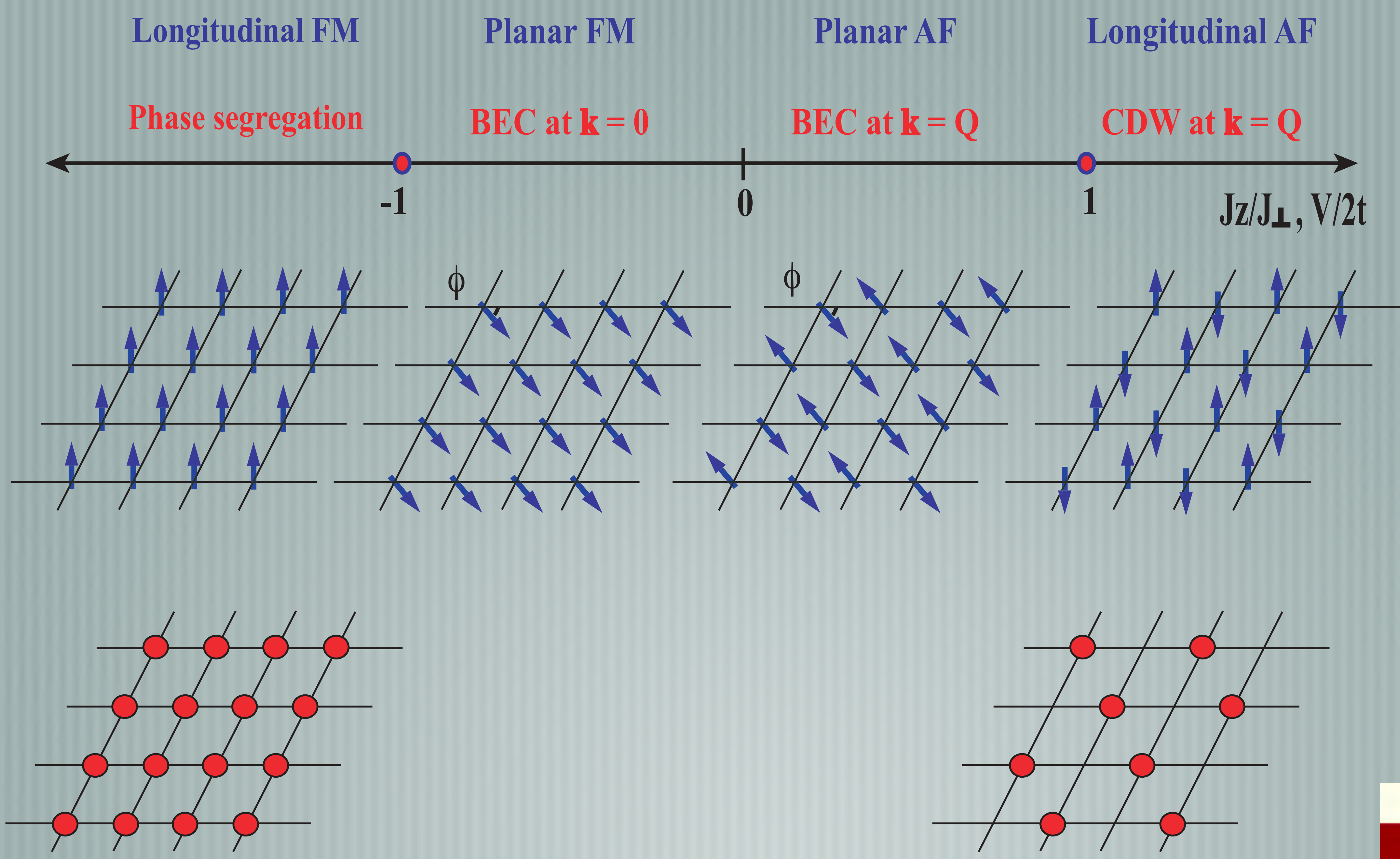
Connect Seemingly Unrelated Physical Phenomena

- Can one map the Physics of a **quantum gas of particles** to that of a **quantum magnet**? Unveiling Order Parameters and Universality



Connect Seemingly Unrelated Physical Phenomena

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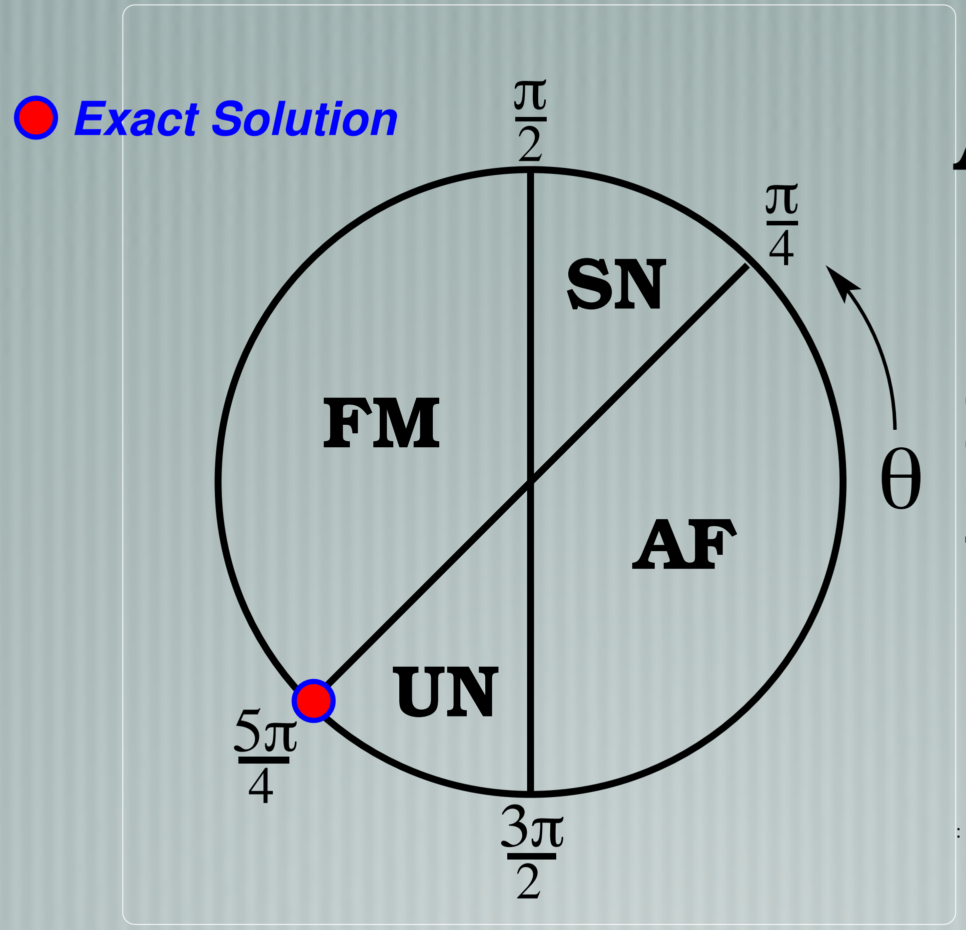
Obtain Exact Solutions



AF: *Antiferromagnetic*

SN: *Staggered–nematic*

FM: *Ferromagnetic*

UN: *Uniform–nematic*

$$H_\theta(J) = J\sqrt{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left[\cos\theta\ \mathbf{S_i}\cdot\mathbf{S_j} + \sin\theta\,(\mathbf{S_i}\cdot\mathbf{S_j})^2\right]$$

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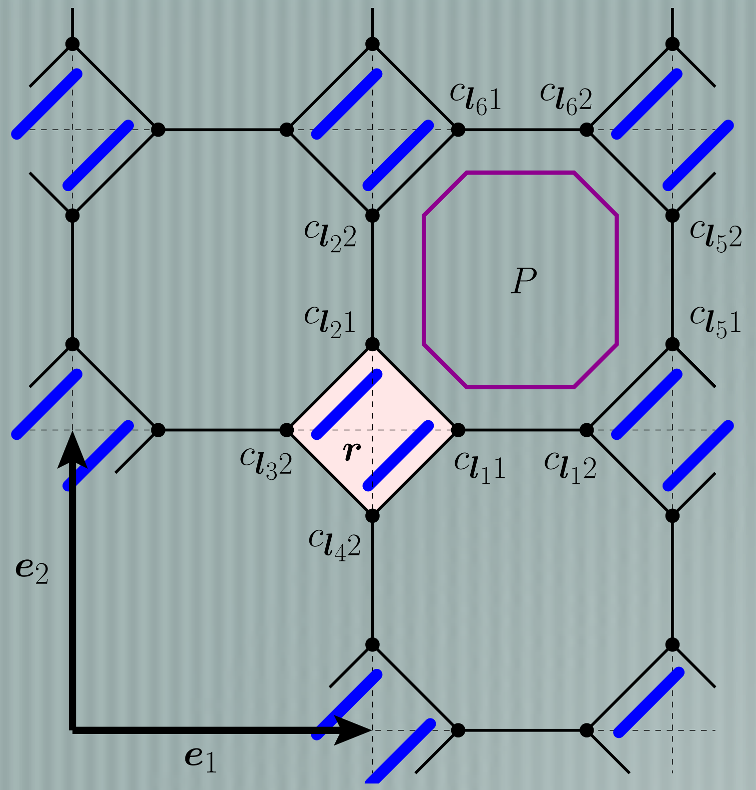
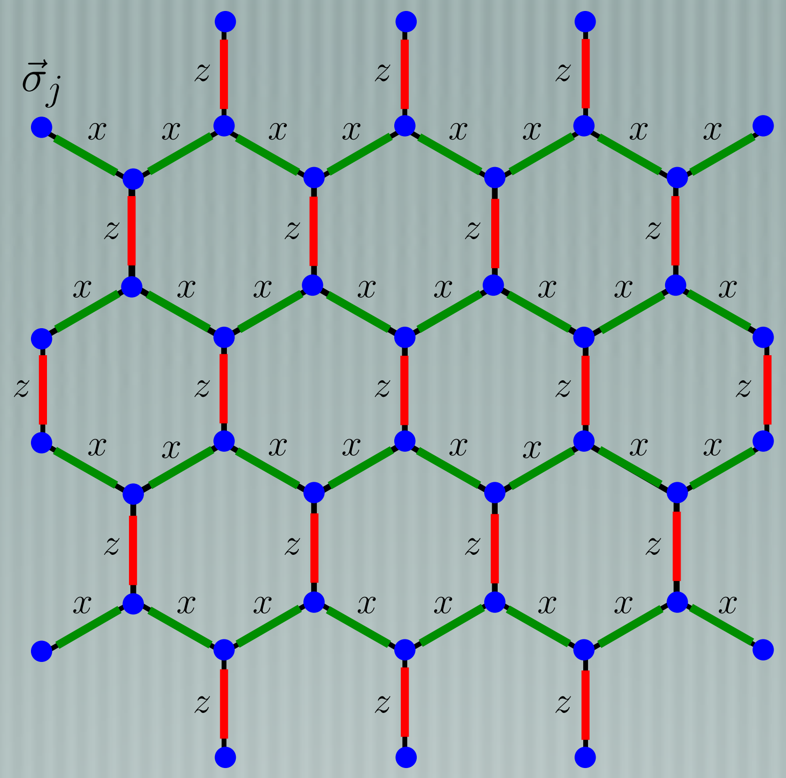
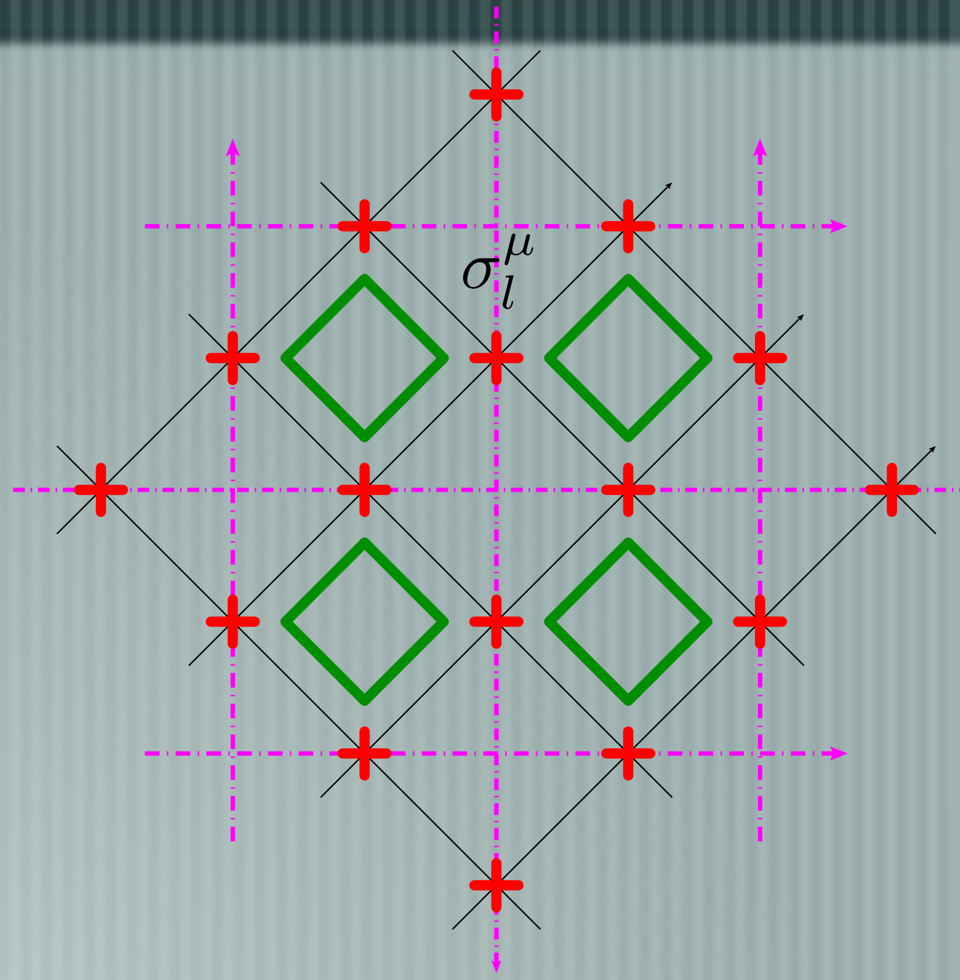
What is the connection? – Duality



Majorana wires
(Interacting)

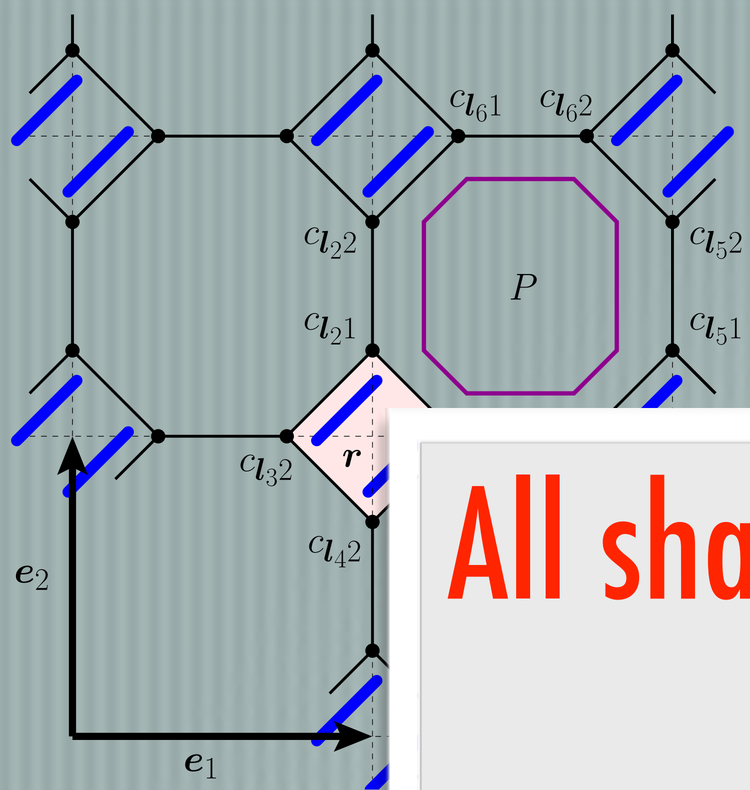


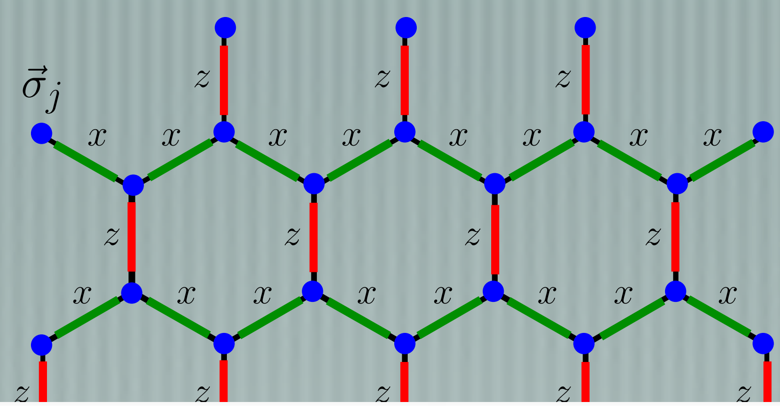
XXZ honeycomb
(spin 1/2)

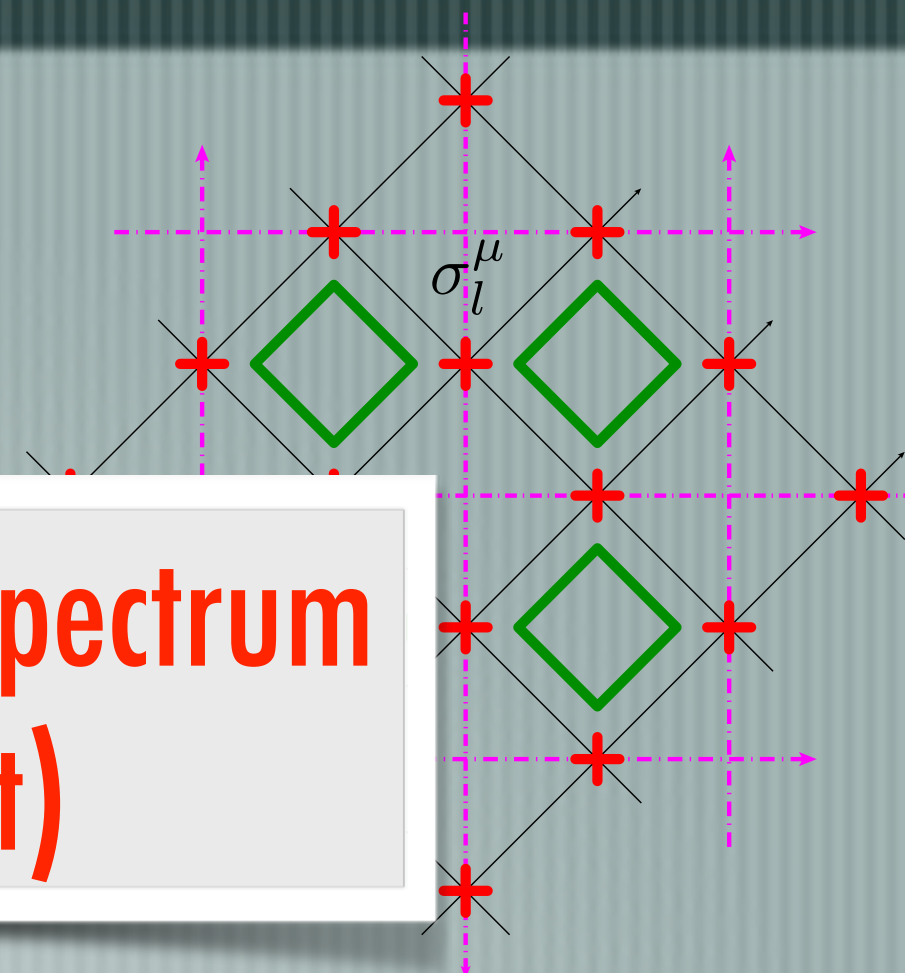


Quantum Ising Gauge

What is the connection? – Duality







All share exactly the same spectrum
*(unitarily equivalent)

Majorana wires
(Interacting)

XXZ honeycomb
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Quantum Ising Gauge

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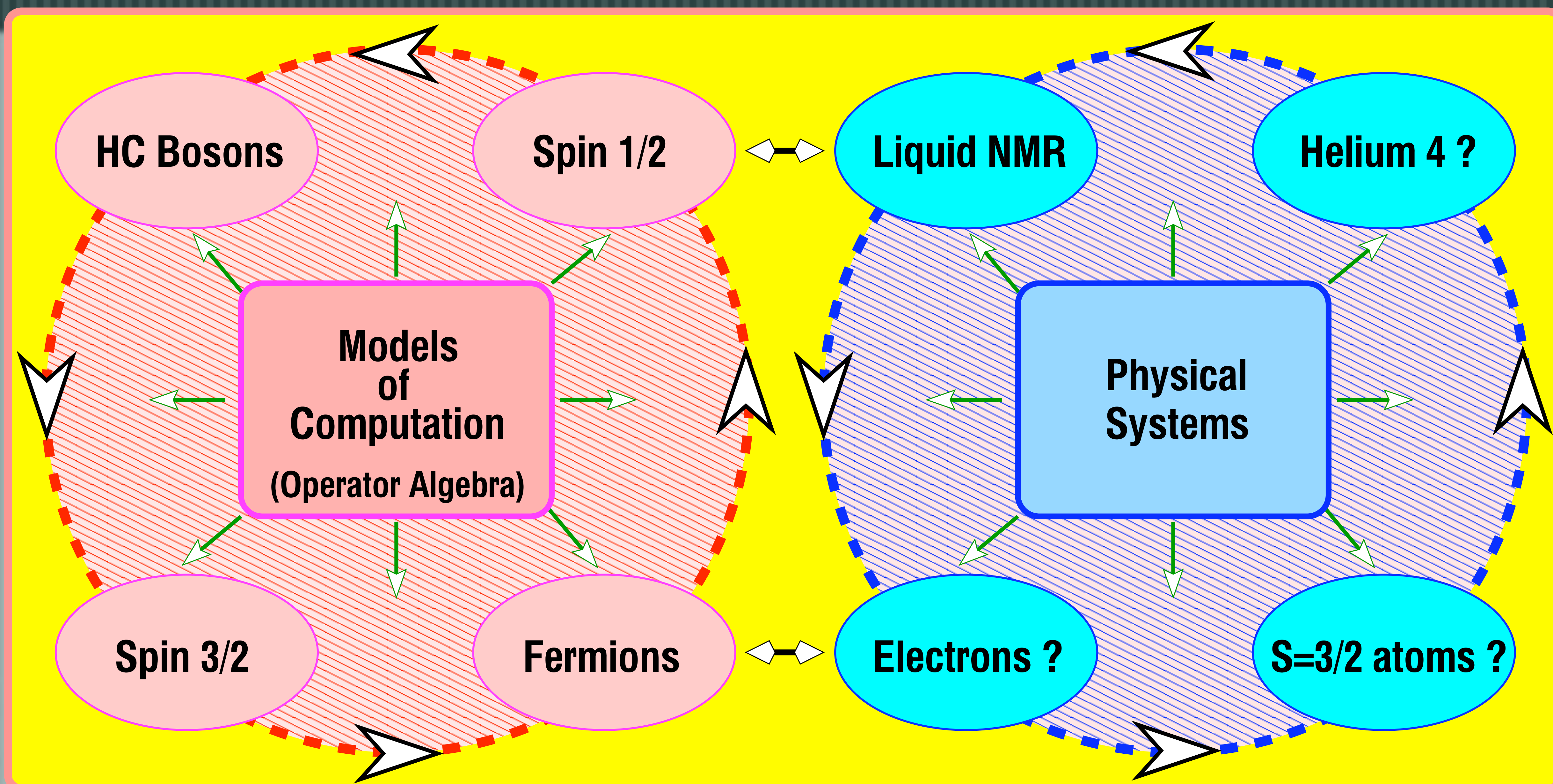
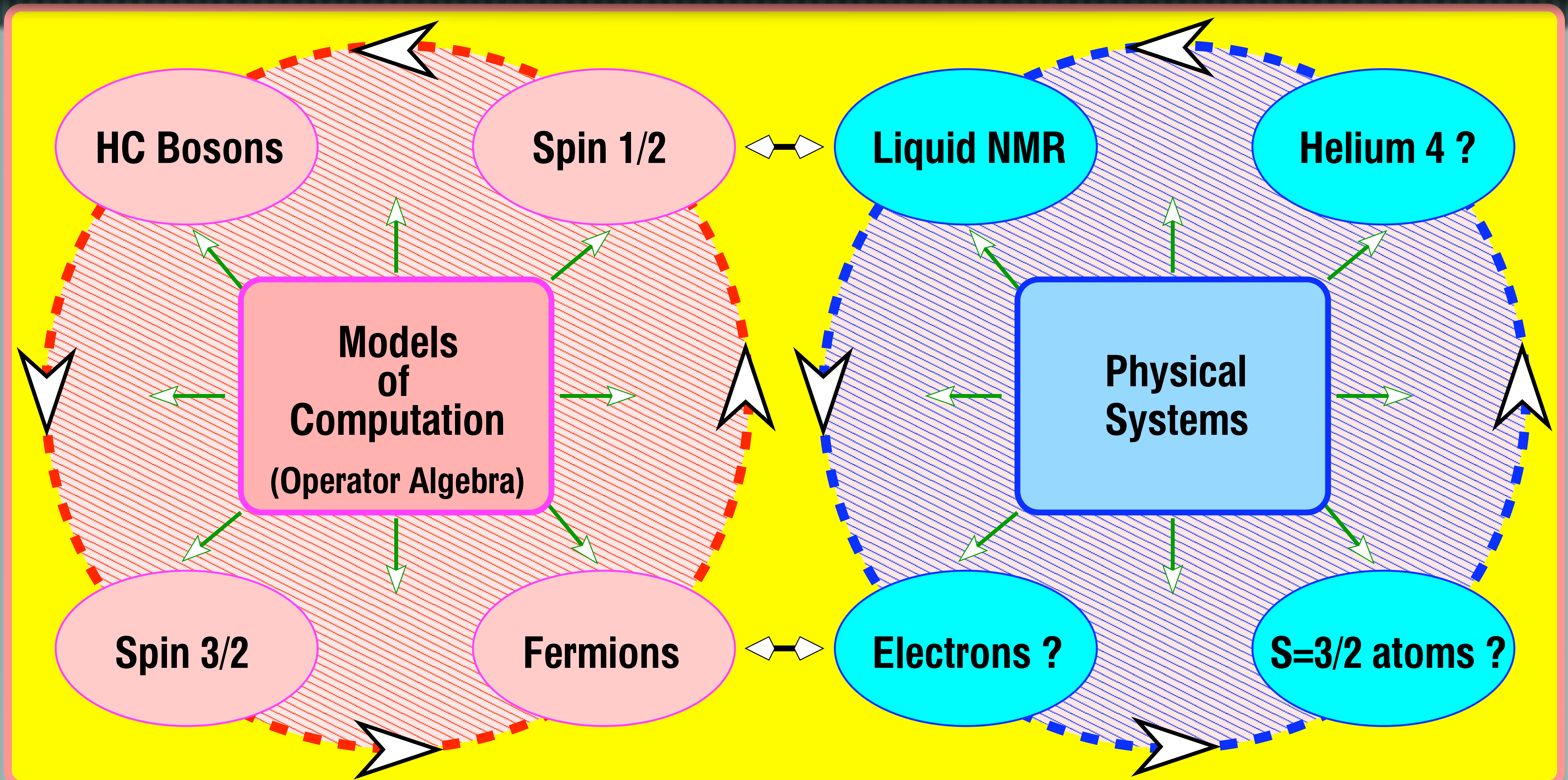
Models of Computation and Physical Systems



A model requires physical systems that can be controlled by modulating the parameters of the system Hamiltonian

The Laws of Computation are the Laws of Physics



A model requires physical systems that can be controlled by modulating the parameters of the system Hamiltonian

OUTLINE

The Many Languages and Dictionaries of Nature:

- Bosonic and Hierarchical Languages
- Transmutation of Statistics: Fermionic (Anyonic) Languages
- Building Dictionaries: Ex. Generalized Jordan-Wigner transformations

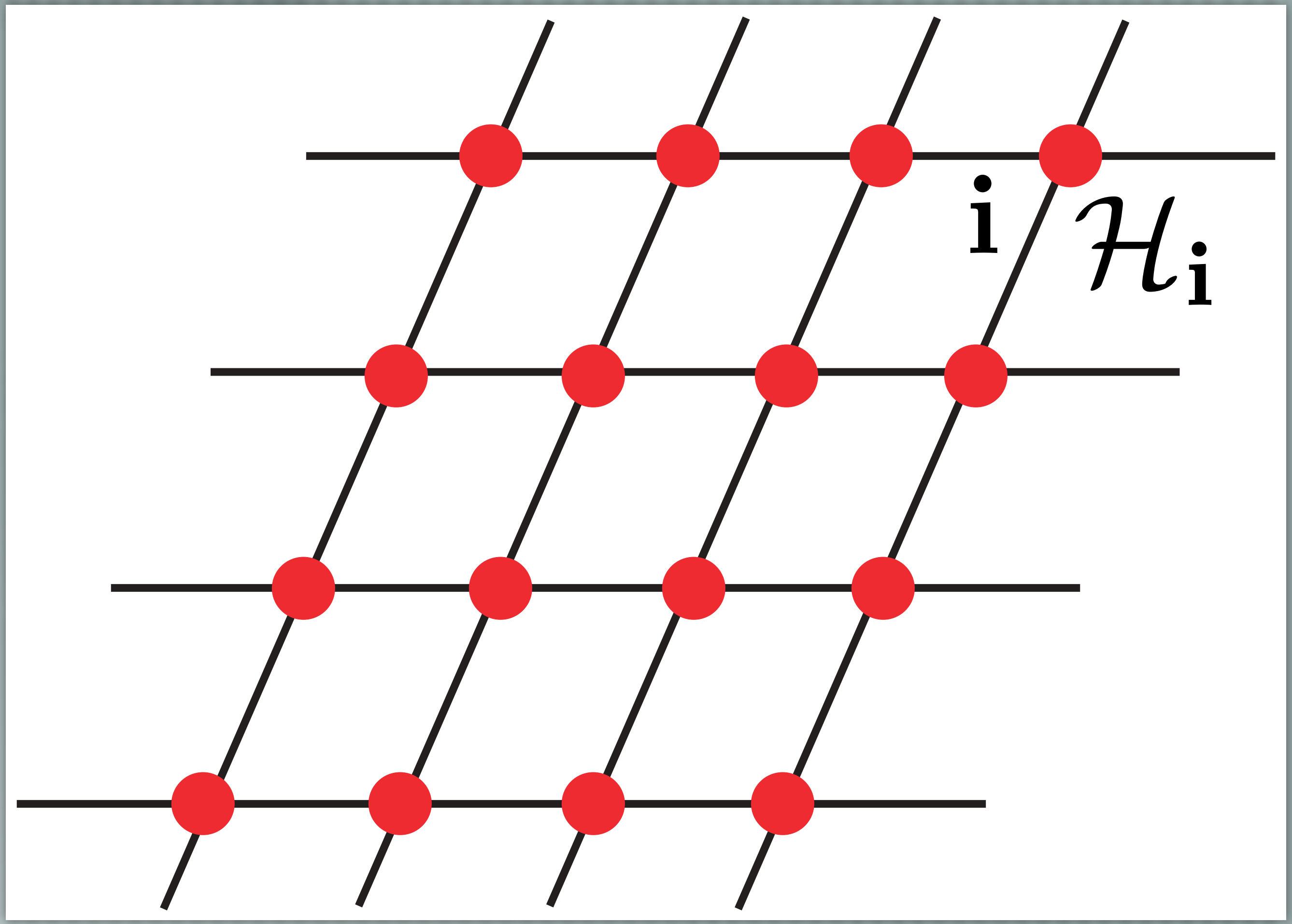
Some Applications:

Unveiling Order and Organizing Principles behind Complexity

Quantum Simulations: Computer vs Simulator

Fermions and Qubits

Problem Setup



- **State Space:** $|\Psi\rangle \in \mathcal{H}$

$$\mathcal{H} = \otimes_{\mathbf{i}} \mathcal{H}_{\mathbf{i}}$$

$$\dim \mathcal{H}_{\mathbf{i}} = D$$

- **Observables:**

$$\hat{\mathcal{O}} \quad (\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}) \qquad \text{Energy, Polarization}, \cdots$$

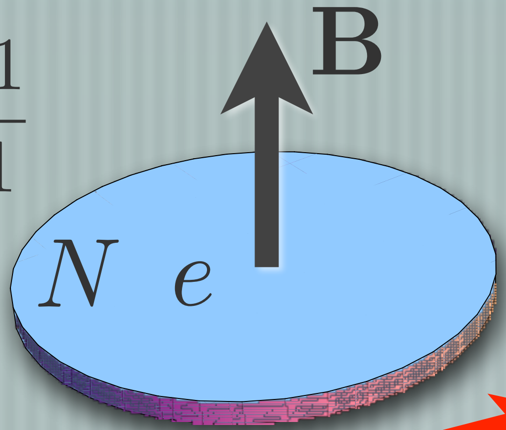
- **Dynamics:** $H = \sum_{\mathbf{i},\mathbf{j}} \langle \mathbf{i}|\mathsf{h}|\mathbf{j}\rangle\, \mathsf{a}^\dagger_{\mathbf{i}} \mathsf{a}_{\mathbf{j}} + \frac{1}{2} \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} \langle \mathbf{ij}|\mathsf{g}|\mathbf{kl}\rangle\, \mathsf{a}^\dagger_{\mathbf{i}} \mathsf{a}^\dagger_{\mathbf{j}} \mathsf{a}_{\mathbf{l}} \mathsf{a}_{\mathbf{k}}$

Dimensional Reduction - QH Physics

First Quantization

Second Quantization

$$\nu = \frac{N-1}{L-1}$$

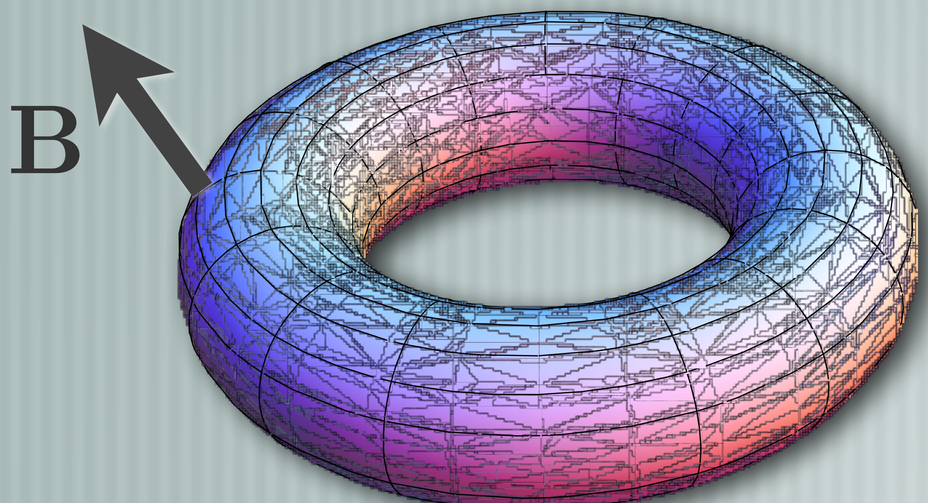


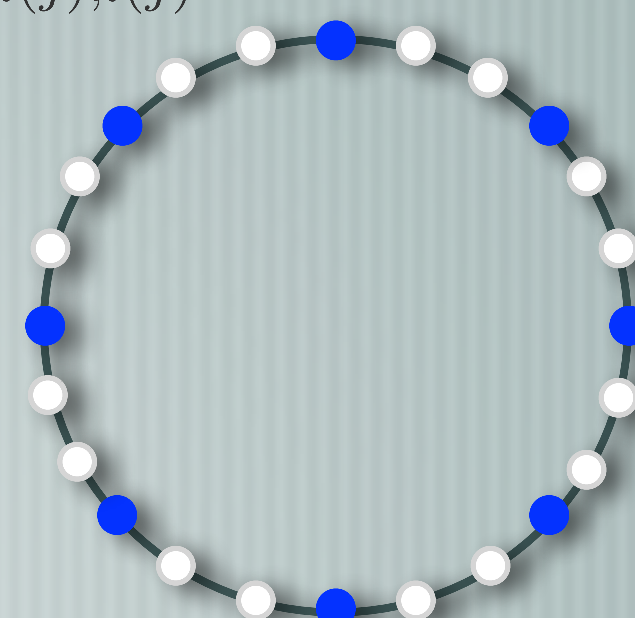
$$\hat{P}_{\mathsf{LLL}} H_{\mathsf{QH}} \hat{P}_{\mathsf{LLL}}$$

dynamical momenta

$$H_{\mathsf{QH}} = \sum_{\mathsf{i}=1}^{N} \frac{\mathbf{\Pi}_{\mathsf{i}}^2}{2m} + \sum_{\mathsf{i}<\mathsf{j}} V(\mathbf{x}_{\mathsf{i}} - \mathbf{x}_{\mathsf{j}})$$

$$\widehat{H}_{\mathsf{QH}} = \sum_{0<j<L-1} \sum_{k(j),l(j)} V_{j;kl}\ c^{\dagger}_{j+k} c^{\dagger}_{j-k} c_{j-l} c_{j+l}$$





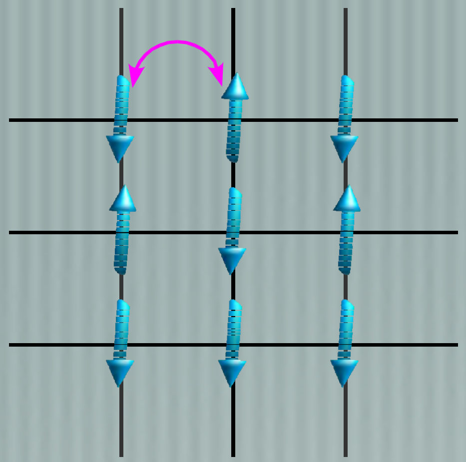
2D continuous geometries

1D orbital lattices

Languages and Dictionaries: Classical Systems

Classical spins to lattice gas or binary alloy mappings

Ising Model: (magnetism)

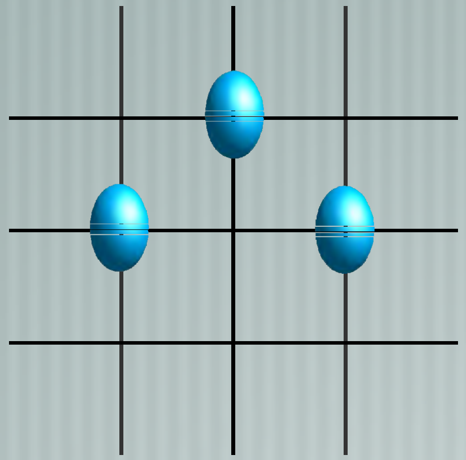


$\{S_i\} \in$ configuration; $S_i = \pm 1$

$$H = J \sum_{\langle i,j \rangle} S_i S_j - B \sum_i S_i$$

$$N_\uparrow + N_\downarrow = N_s$$

Lattice Gas: (e.g., crystal melting)

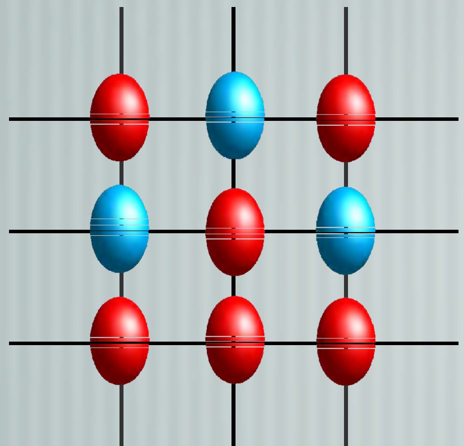


Collection of "atoms" (no T)

One atom at each site

$N_\uparrow \to N_{\color{blue}\bullet}$, $N_\downarrow \to$ empty sites

Binary Alloy: (e.g., β-brass)



Two types of "atoms" (no T)

One atom at each site (${\color{blue}\bullet}$ or ${\color{red}\bullet}$)

$$N_{\color{blue}\bullet} + N_{\color{red}\bullet} = N_s$$

Languages and Dictionaries: Quantum Systems

A bosonic language is a set of operators grouped in subsets $S_{\mathbf{i}}$ (associated to each mode $\mathbf{i}$) that satisfy the conditions:

- Each $S_{\mathbf{i}}$ is a set of elements $\mathfrak{b}_{\mathbf{i}}^{\mu}$ which belong to the algebra of endomorphisms for the vector space $\mathcal{H}_{\mathbf{i}}$ over the field $\mathbb{C}$, $\mathfrak{b}_{\mathbf{i}}^{\mu} : \mathcal{H}_{\mathbf{i}} \to \mathcal{H}_{\mathbf{i}}$, and are linearly independent.

- The elements of $S_{\mathbf{i}}$ generate a monoid of linear transformations under the associative product in the algebra which acts irreducibly on $\mathcal{H}_{\mathbf{i}}$ in the sense that the only subspaces stabilized by $S_{\mathbf{i}}$ are $\mathcal{H}_{\mathbf{i}}$ and $\emptyset$.

- If $\mathfrak{b}_{\mathbf{i}}^{\mu}$ and $\mathfrak{b}_{\mathbf{j}}^{\nu}$ are elements of different subsets $S_{\mathbf{i}}$ and $S_{\mathbf{j}}$, then

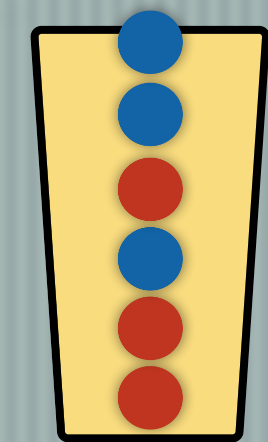
$$[\mathfrak{b}_{\mathbf{i}}^{\mu}, \mathfrak{b}_{\mathbf{j}}^{\nu}] = \mathfrak{b}_{\mathbf{i}}^{\mu}\mathfrak{b}_{\mathbf{j}}^{\nu} - \mathfrak{b}_{\mathbf{j}}^{\nu}\mathfrak{b}_{\mathbf{i}}^{\mu} = 0 \ , \quad \text{if } \ \mathbf{i} \neq \mathbf{j}$$

Bosonic Languages

$1 \leq \alpha,\ \beta \leq N_f$

$N_f = 2$

$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} \ , \ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$



i

CCR

$D \to \infty$

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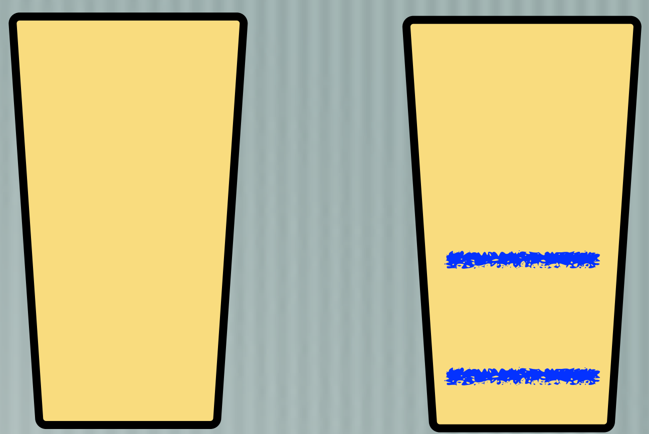

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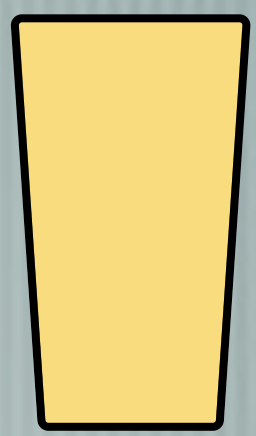

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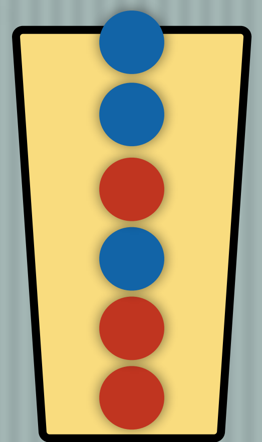

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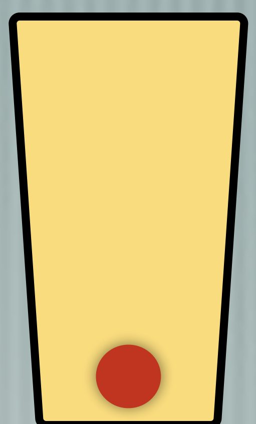

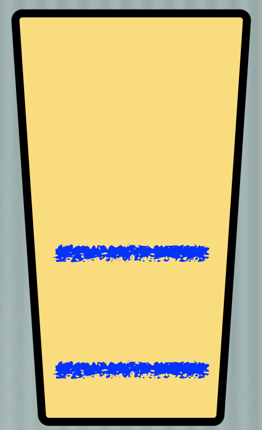

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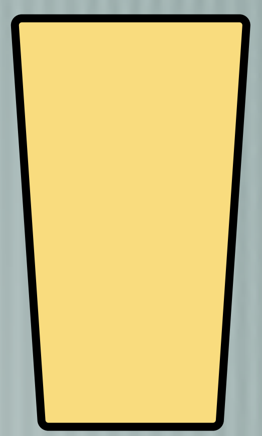

Bosonic Languages

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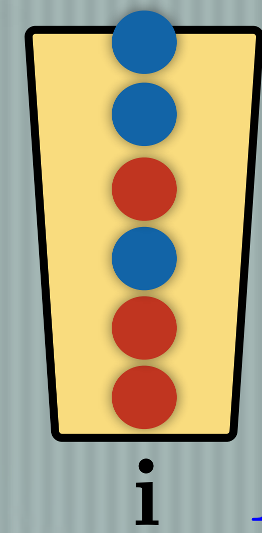

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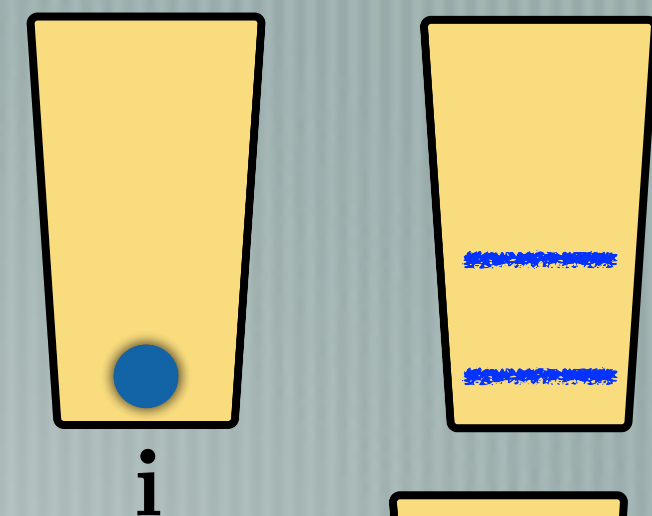

$D = N_f + 1$

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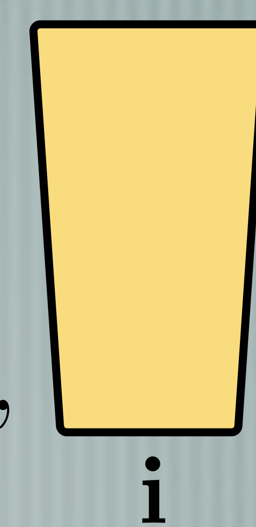

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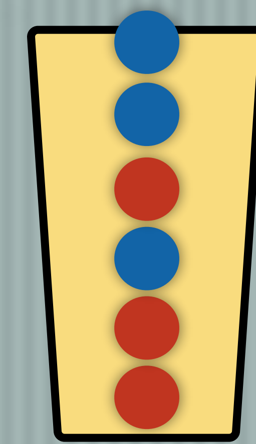

$\mathbf{i}$

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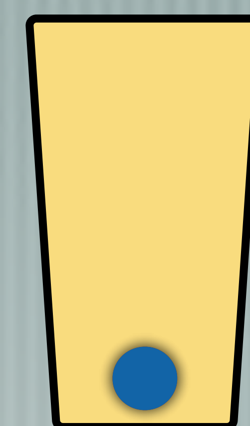
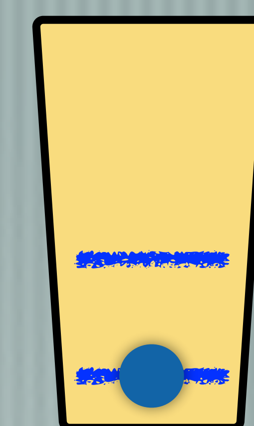
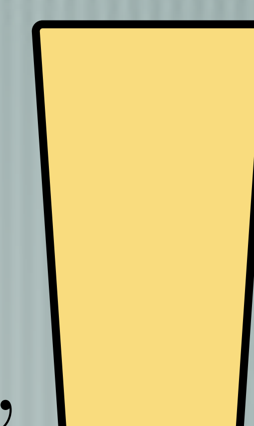




$\mathbf{i}$

$D = N_f + 1$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_{\alpha}) \ , \ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$

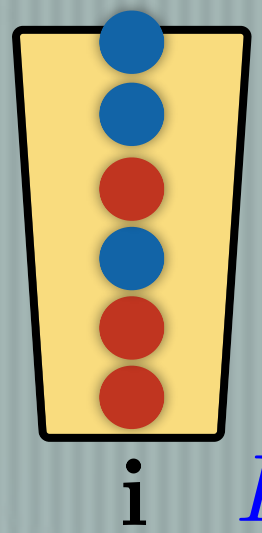


$\mathbf{i}$

Bosonic Languages

$1 \leq \alpha,\ \beta \leq N_f$ $\qquad N_f = 2$

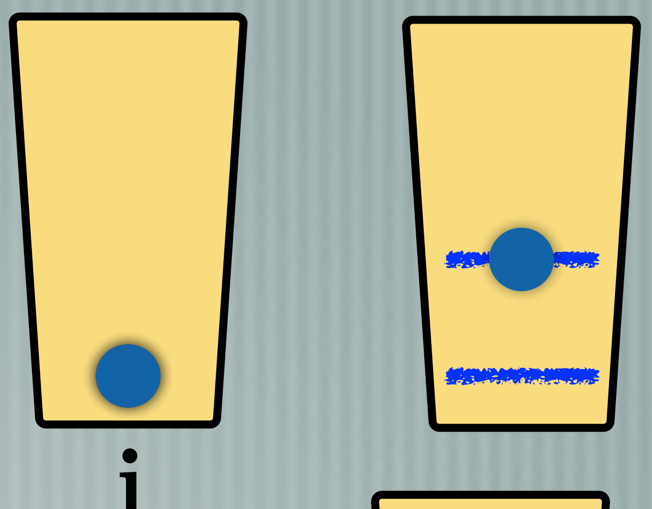
$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\ ,\ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$



CCR

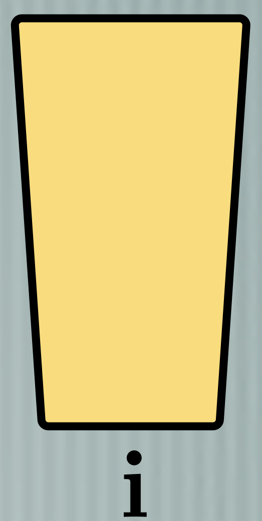
$D \to \infty$

$$\begin{cases} [\bar{b}_{\mathbf{i}\alpha}, \bar{b}_{\mathbf{j}\beta}] = [\bar{b}^\dagger_{\mathbf{i}\alpha}, \bar{b}^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\alpha}] = \delta_{\mathbf{ij}}(\delta_{\alpha\beta} - \bar{n}_{\mathbf{i}}\delta_{\alpha\beta} - \bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta})\ , \\ [\bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\gamma}] = \delta_{\mathbf{ij}}\delta_{\beta\gamma}\bar{b}^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$



$D = N_f + 1$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_\alpha)\ ,\ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$

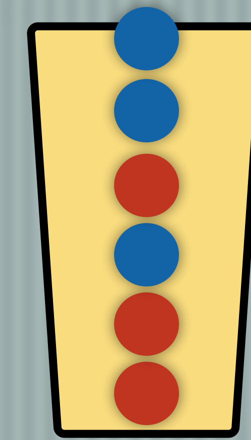


Bosonic Languages

$1 \leq \alpha,\ \beta \leq N_f$

$N_f = 2$

$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} \ , \ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$

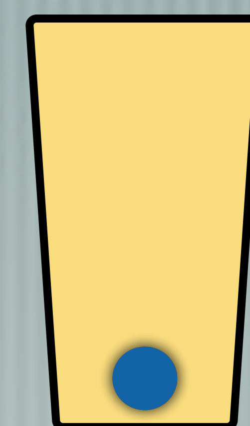


$\mathbf{i}$

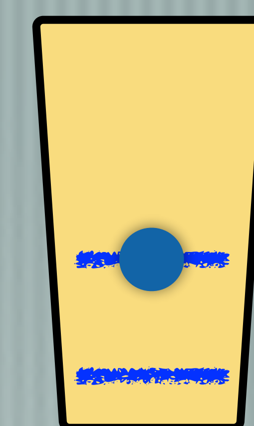
CCR

$D \to \infty$

$$\begin{cases} [\bar{b}_{\mathbf{i}\alpha}, \bar{b}_{\mathbf{j}\beta}] = [\bar{b}^\dagger_{\mathbf{i}\alpha}, \bar{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\alpha}] = \delta_{\mathbf{ij}}(\delta_{\alpha\beta} - \bar{n}_{\mathbf{i}}\delta_{\alpha\beta} - \bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}) \ , \\ [\bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\gamma}] = \delta_{\mathbf{ij}}\delta_{\beta\gamma}\bar{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$

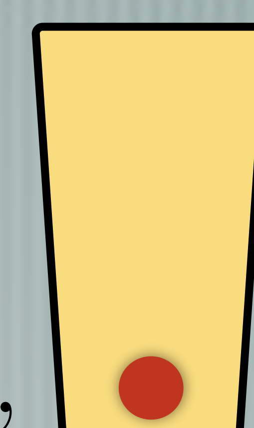


$\mathbf{i}$



$D = N_f + 1$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_\alpha) \ , \ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$



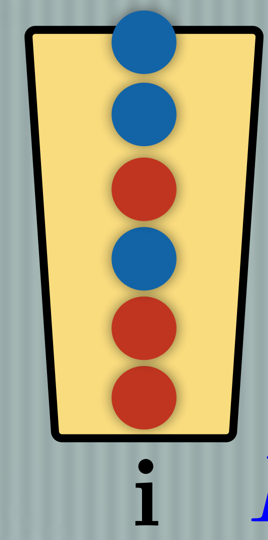
$\mathbf{i}$

Bosonic Languages

$1 \le \alpha,\ \beta \le N_f$

$N_f = 2$

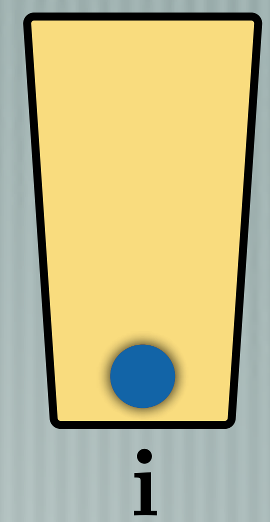
$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} \ , \ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$

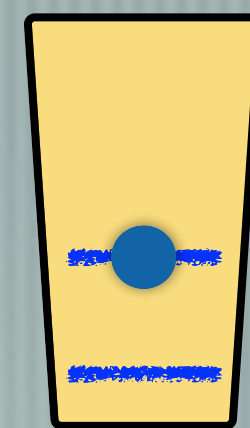


CCR

$D \to \infty$

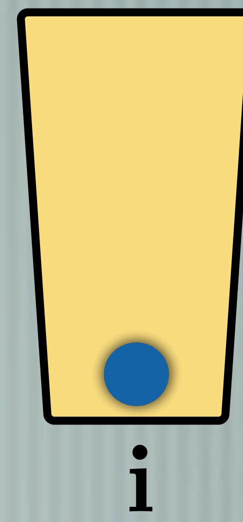
$$\begin{cases} [\bar{b}_{\mathbf{i}\alpha}, \bar{b}_{\mathbf{j}\beta}] = [\bar{b}^\dagger_{\mathbf{i}\alpha}, \bar{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\alpha}] = \delta_{\mathbf{ij}}(\delta_{\alpha\beta} - \bar{n}_{\mathbf{i}}\delta_{\alpha\beta} - \bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}) \ , \\ [\bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\gamma}] = \delta_{\mathbf{ij}}\delta_{\beta\gamma}\bar{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$





$D = N_f + 1$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_\alpha) \ , \ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$

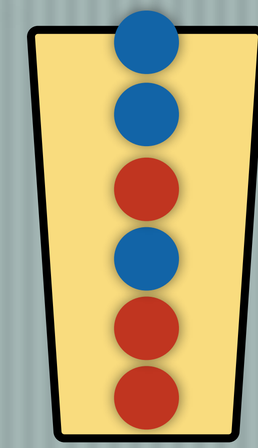


Bosonic Languages

$1 \leq \alpha,\ \beta \leq N_f$

$N_f = 2$

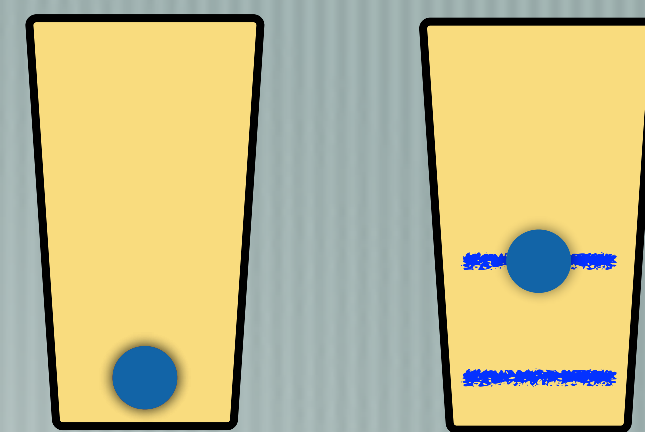
$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} \ , \ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$



CCR

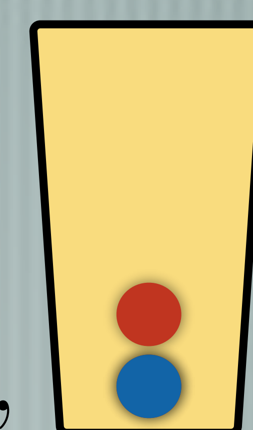
$\mathbf{i}$ $D \to \infty$

$$\begin{cases} [\bar{b}_{\mathbf{i}\alpha}, \bar{b}_{\mathbf{j}\beta}] = [\bar{b}^\dagger_{\mathbf{i}\alpha}, \bar{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\alpha}] = \delta_{\mathbf{ij}}(\delta_{\alpha\beta} - \bar{n}_{\mathbf{i}}\delta_{\alpha\beta} - \bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}) \ , \\ [\bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\gamma}] = \delta_{\mathbf{ij}}\delta_{\beta\gamma}\bar{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$



$D = N_f + 1$ $\mathbf{i}$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0 \ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_\alpha) \ , \ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha} \ , \end{cases}$$



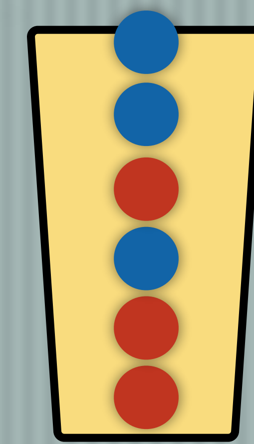
$\mathbf{i}$ $D = 2^{N_f}$

Bosonic Languages

$1 \leq \alpha,\ \beta \leq N_f$

$N_f = 2$

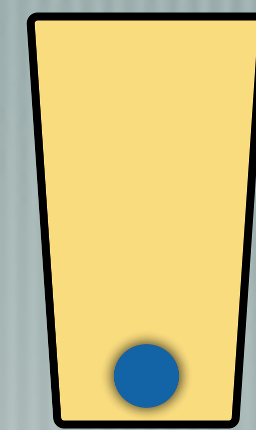
$$\begin{cases} [b_{\mathbf{i}\alpha}, b_{\mathbf{j}\beta}] = [b^\dagger_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [b_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\ ,\ [n_{\mathbf{i}\alpha}, b^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta} b^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$

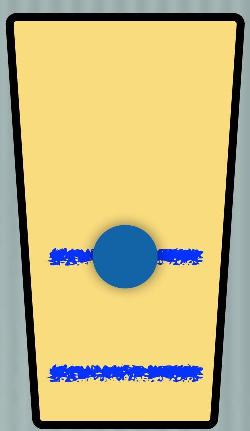


CCR

$\mathbf{i}$ $\quad D \to \infty$

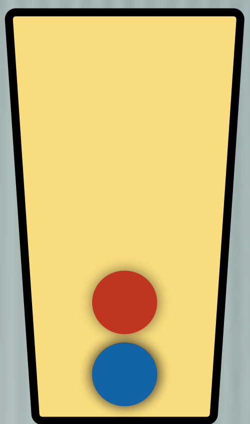
$$\begin{cases} [\bar{b}_{\mathbf{i}\alpha}, \bar{b}_{\mathbf{j}\beta}] = [\bar{b}^\dagger_{\mathbf{i}\alpha}, \bar{b}^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\alpha}] = \delta_{\mathbf{ij}}(\delta_{\alpha\beta} - \bar{n}_{\mathbf{i}}\delta_{\alpha\beta} - \bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta})\ , \\ [\bar{b}^\dagger_{\mathbf{i}\alpha}\bar{b}_{\mathbf{i}\beta}, \bar{b}^\dagger_{\mathbf{j}\gamma}] = \delta_{\mathbf{ij}}\delta_{\beta\gamma}\bar{b}^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$





$D = N_f + 1$ $\quad \mathbf{i}$

$$\begin{cases} [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}_{\mathbf{j}\beta}] = [\tilde{b}^\dagger_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = 0\ , \\ [\tilde{b}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}(1 - 2\tilde{n}_\alpha)\ ,\ [\tilde{n}_{\mathbf{i}\alpha}, \tilde{b}^\dagger_{\mathbf{j}\beta}] = \delta_{\mathbf{ij}}\delta_{\alpha\beta}\tilde{b}^\dagger_{\mathbf{i}\alpha}\ , \end{cases}$$



$\mathbf{i}$ $\quad D = 2^{N_f}$

Physics defines what language is more appropriate

Fundamental Theorem

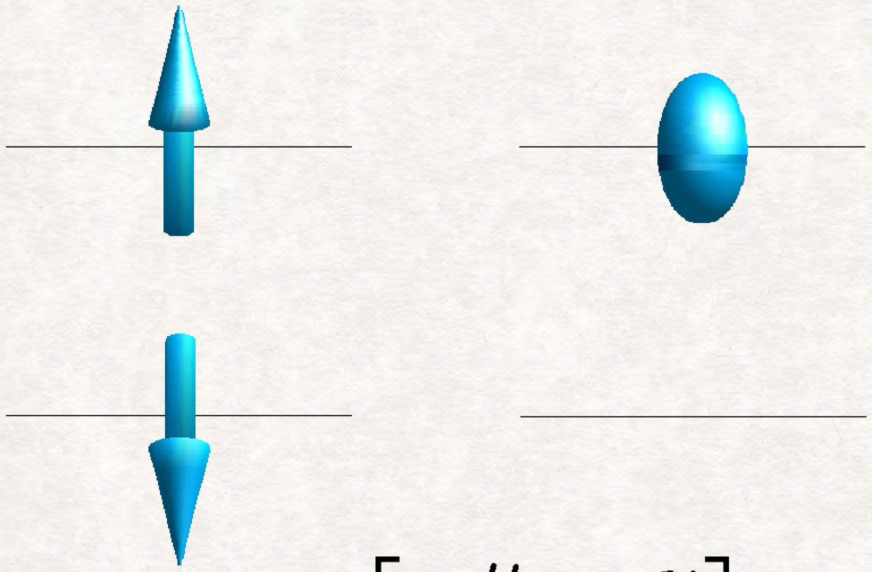
Given two bosonic languages having the same finite dimension D of their local Hilbert spaces $\mathcal{H}_{\mathbf{i}}$, the generators of one of them can can be written as a polynomial function of the generators of the other language and vice versa.

(Corollary of Burnside's Theorem)

Languages and Dictionaries

Example I: Matsubara-Matsuda transformation.

Spin 1/2 Hard Core Boson



$$
\begin{aligned}
S_j^+ &= \bar{b}_j^\dagger \\
S_j^- &= \bar{b}_j \\
S_j^z &= \bar{b}_j^\dagger \bar{b}_j - \frac{1}{2}
\end{aligned}
$$

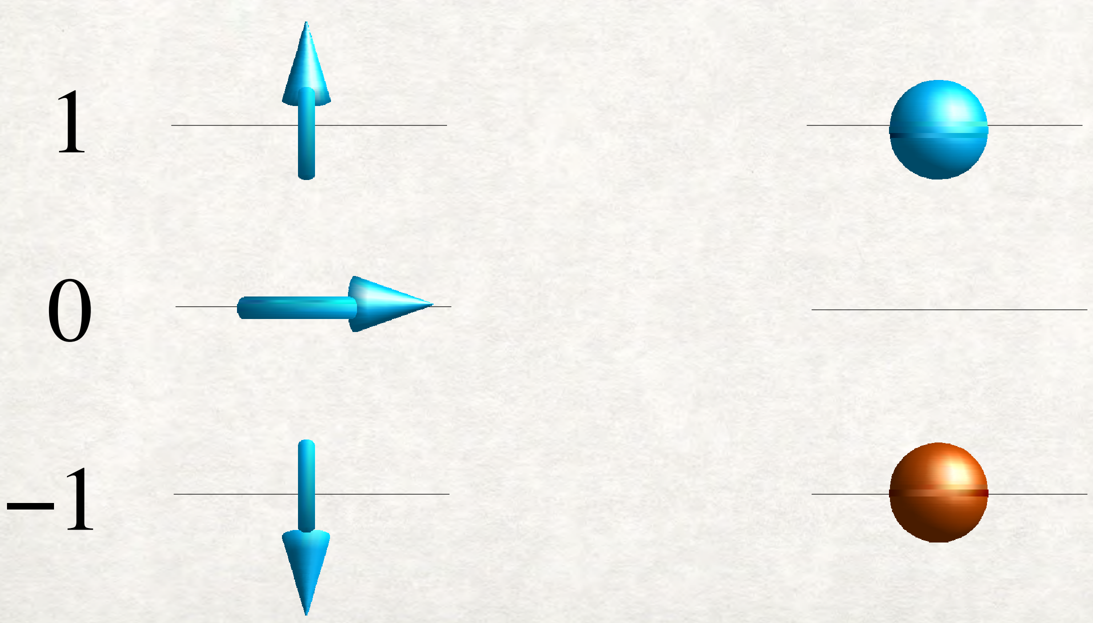
$$
[S_{\mathbf{i}}^\mu, S_{\mathbf{j}}^\nu] = i\delta_{\mathbf{ij}}\epsilon_{\mu\nu\lambda}S_{\mathbf{i}}^\lambda \,, \quad \mu, \nu, \lambda = x, y, z
$$

Languages and Dictionaries

Example II: Generalization of the Matsubara-Matsuda transformation to $S = 1$

SU(2) Spin One

S=1/2 hard core bosons

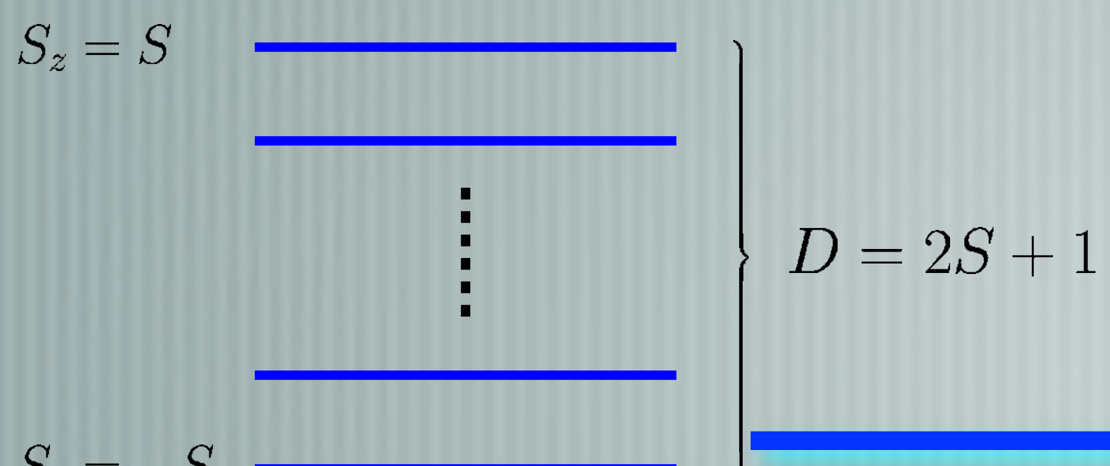


$$
\begin{aligned}
S_j^+ &= \sqrt{2}\,(\bar{b}^\dagger_{j\uparrow} + \bar{b}_{j\downarrow}) \\
S_j^- &= \sqrt{2}\,(\bar{b}_{j\uparrow} + \bar{b}^\dagger_{j\downarrow}) \\
S_j^z &= \bar{n}_{j\uparrow} - \bar{n}_{j\downarrow}
\end{aligned}
$$

Fundamental Theorem

Given two bosonic languages having the same finite dimension D of their local Hilbert spaces $\mathcal{H}_{\mathbf{i}}$, the generators of one of them can can be written as a polynomial function of the generators of the other language and vice versa. (Corollary of Burnside's Theorem)

Corollary: In each class of bosonic languages there is at least one which is the conjunction of a Lie algebra $\mathcal{S}$ and an irreducible rep $\Gamma_{\mathcal{S}}$ ($\mathcal{S} \wedge \Gamma_{\mathcal{S}}$), i.e., the generators of the bosonic language are generators of the Lie algebra $\mathcal{S}_{\mathbf{i}}$ in the rep $\Gamma_{\mathcal{S}}$.



$$\mathcal{L}_{\mathbf{i}} = u(1) \bigoplus su(2)$$

$$\mathcal{S} = \bigoplus_{\mathbf{i}} \mathcal{L}_{\mathbf{i}}$$

Hierarchical Language

- **Definition:** **Any local operator $\hat{\mathcal{O}}$ can be written as a linear combination of the generators of the language.**

$$\hat{\mathcal{O}} = \sum_{i=0}^{N_G} \lambda_i \, G_i$$

- **For each class of languages there is always one hierarchical language (HL) whose generators are the identity I and the generators of $su(N)$ in the fundamental representation ($D = N$).**

- **A Hamiltonian operator in the HL becomes quadratic in the symmetry generators of the Hierarchical group.**

- **Example:** $N = 2$ **(Two-level system). The Pauli matrices are generators of $su(2)$ in the fundamental representation.**

$$\hat{\mathcal{O}} = \lambda_0 \, I + \lambda_1 \, \sigma_i^x + \lambda_2 \, \sigma_i^y + \lambda_3 \, \sigma_i^z$$

SU(N) Spin-Particle Mappings

$$\mathcal{S}^{\alpha\beta}(\mathbf{j}) = \bar{b}^{\dagger}_{\mathbf{j}\alpha}\bar{b}_{\mathbf{j}\beta} - \frac{\delta_{\alpha\beta}}{N}$$

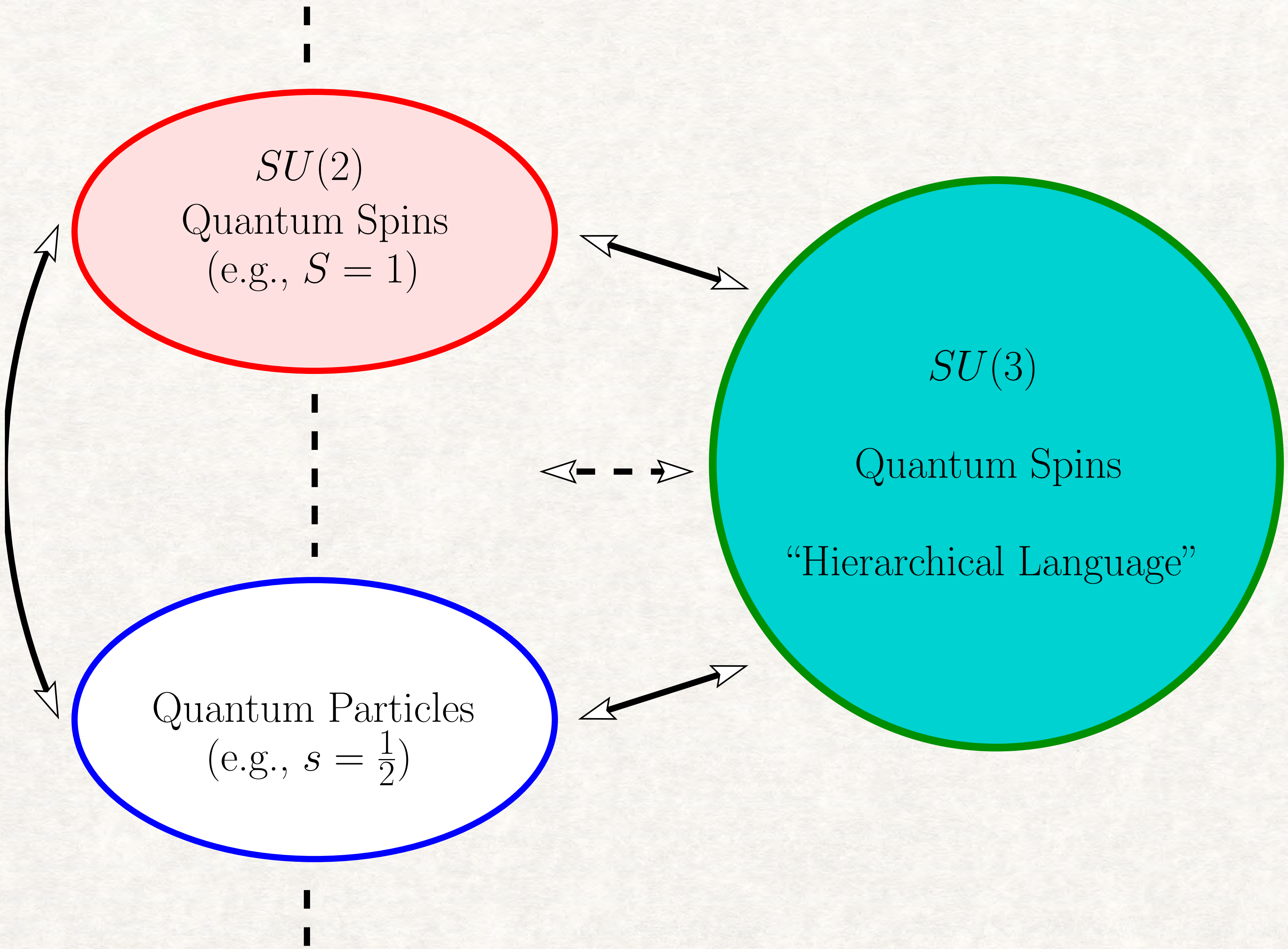
$$\mathcal{S}^{\alpha 0}(\mathbf{j}) = \bar{b}^{\dagger}_{\mathbf{j}\alpha} \ , \quad \mathcal{S}^{0\beta}(\mathbf{j}) = \bar{b}_{\mathbf{j}\beta}$$

$$\mathcal{S}^{00}(\mathbf{j}) = \frac{N_f}{N} - \sum_{\alpha=1}^{N_f} \bar{n}_{\mathbf{j}\alpha} = - \sum_{\alpha=1}^{N_f} \mathcal{S}^{\alpha\alpha}(\mathbf{j}) \ ,$$

where $1 \leq \alpha, \beta \leq N_f$ runs over the set of particle flavors.

$$[\mathcal{S}^{\mu\mu'}(\mathbf{j}), \mathcal{S}^{\nu\nu'}(\mathbf{j})] = \delta_{\mu'\nu}\mathcal{S}^{\mu\nu'}(\mathbf{j}) - \delta_{\mu\nu'}\mathcal{S}^{\nu\mu'}(\mathbf{j}).$$

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_{\mathbf{j}} & \bar{b}_{\mathbf{j}1} & \bar{b}_{\mathbf{j}2} \\ \bar{b}^{\dagger}_{\mathbf{j}1} & \bar{n}_{\mathbf{j}1} - \frac{1}{3} & \bar{b}^{\dagger}_{\mathbf{j}1}\bar{b}_{\mathbf{j}2} \\ \bar{b}^{\dagger}_{\mathbf{j}2} & \bar{b}^{\dagger}_{\mathbf{j}2}\bar{b}_{\mathbf{j}1} & \bar{n}_{\mathbf{j}2} - \frac{1}{3} \end{pmatrix}$$


$SU(2)$
Quantum Spins
(e.g., $S = 1$)
Quantum Particles
(e.g., $s = \frac{1}{2}$)
$SU(3)$
Quantum Spins
"Hierarchical Language"

Fermionic Languages

$$\{A, B\} = AB + BA$$

We will say that a fermionic language is the one generated by the creation and anihilation operators for the canonical fermions:

$$\begin{cases} \{c_{\mathbf{i}\alpha}, c_{\mathbf{j}\beta}\} = \{c^{\dagger}_{\mathbf{i}\alpha}, c^{\dagger}_{\mathbf{j}\beta}\} = 0 \ , \\ \{c_{\mathbf{i}\alpha}, c^{\dagger}_{\mathbf{j}\beta}\} = \delta_{\mathbf{ij}}\delta_{\alpha\beta} \end{cases}$$

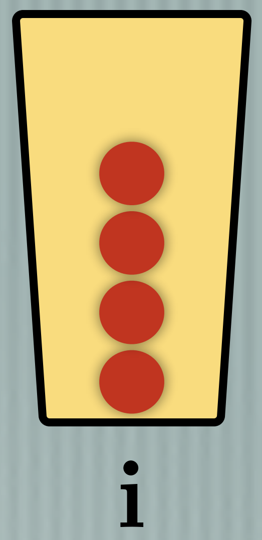
CAR

and any other one which can be obtained by imposing local constraints to the canonical fermions.

Quantum Statistics

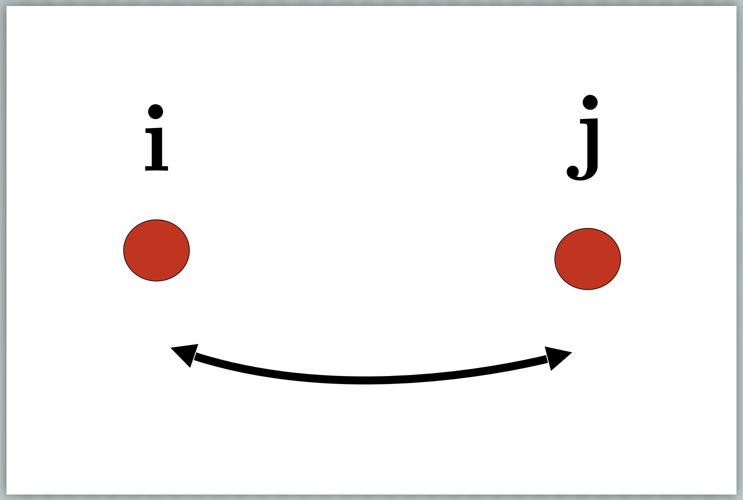
Local Pauli Exclusion Principle:

Fractional Exclusion Statistics parameter p



$p = 4$

Non-local Exchange (Permutation/Braid) Statistics:



Transmutation of Statistics

$\theta = 0$ "Bosons" $\qquad$ $\theta = \pi$ "Fermions"

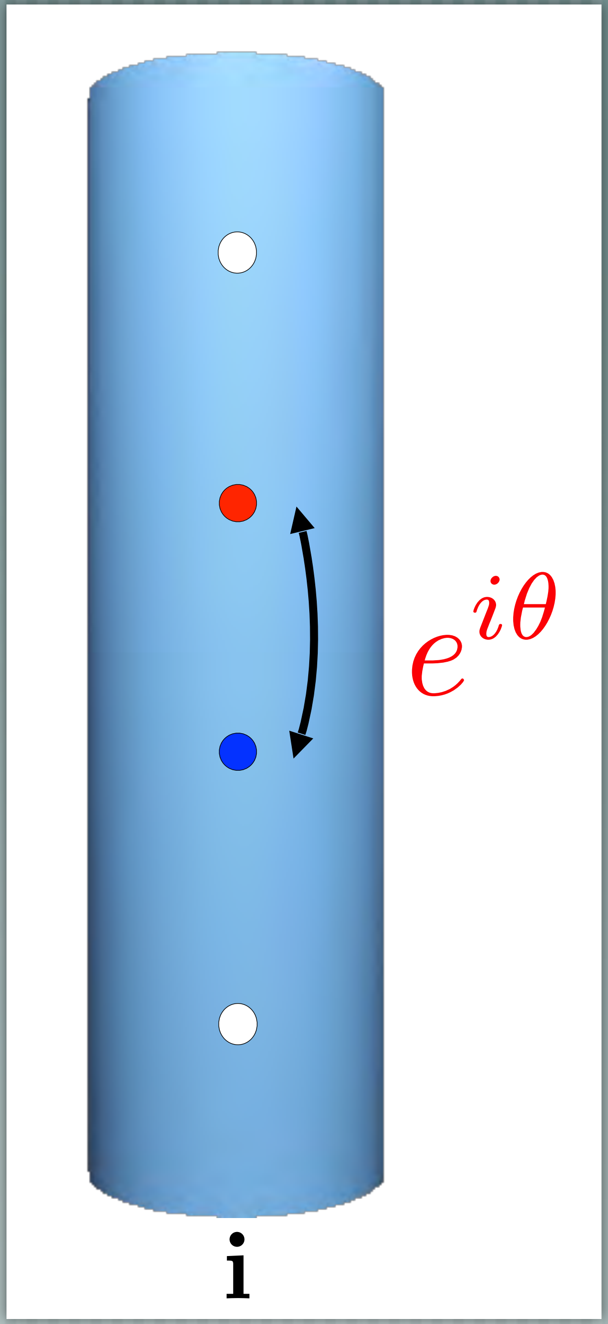
Local Transmutation: $c^\dagger_{\mathbf{j}\alpha} = \tilde{b}^\dagger_{\mathbf{j}\alpha}\hat{\mathcal{T}}^\dagger_{\mathbf{j}\alpha}$,

$$\hat{\mathcal{T}}^\theta_{\mathbf{j}\alpha} = \exp[i\theta \sum_{\beta<\alpha} \tilde{n}_{\mathbf{j}\beta}] \qquad \hat{\mathcal{T}}^{\theta=\pi}_{\mathbf{j}\alpha} = \hat{\mathcal{T}}_{\mathbf{j}\alpha} \qquad \hat{\mathcal{T}}^2_{\mathbf{j}\alpha} = \mathbb{1}, \quad \hat{\mathcal{T}}^\dagger_{\mathbf{j}\alpha} = \hat{\mathcal{T}}_{\mathbf{j}\alpha}$$

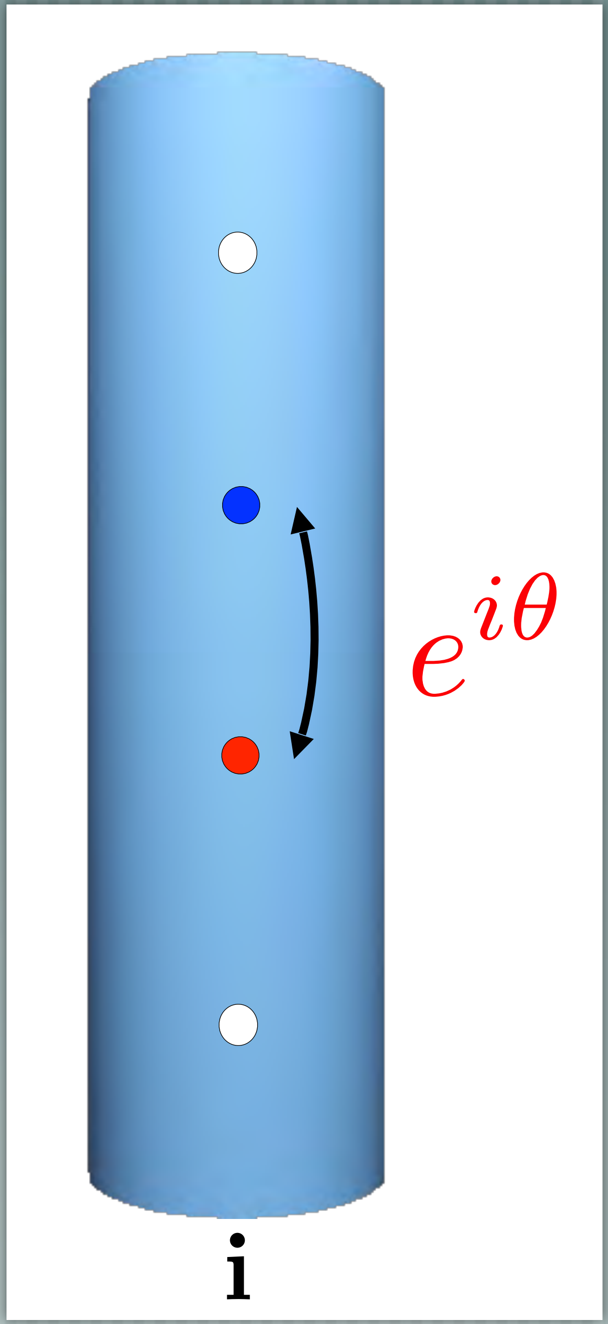
Non-local Transmutation: $c^\dagger_{\mathbf{j}\alpha} = \tilde{b}^\dagger_{\mathbf{j}\alpha}\hat{\mathcal{T}}^\dagger_{\mathbf{j}\alpha}K^\dagger_{\mathbf{j}} = \tilde{b}^\dagger_{\mathbf{j}\alpha}\mathcal{K}^\dagger_{\mathbf{j}\alpha}$

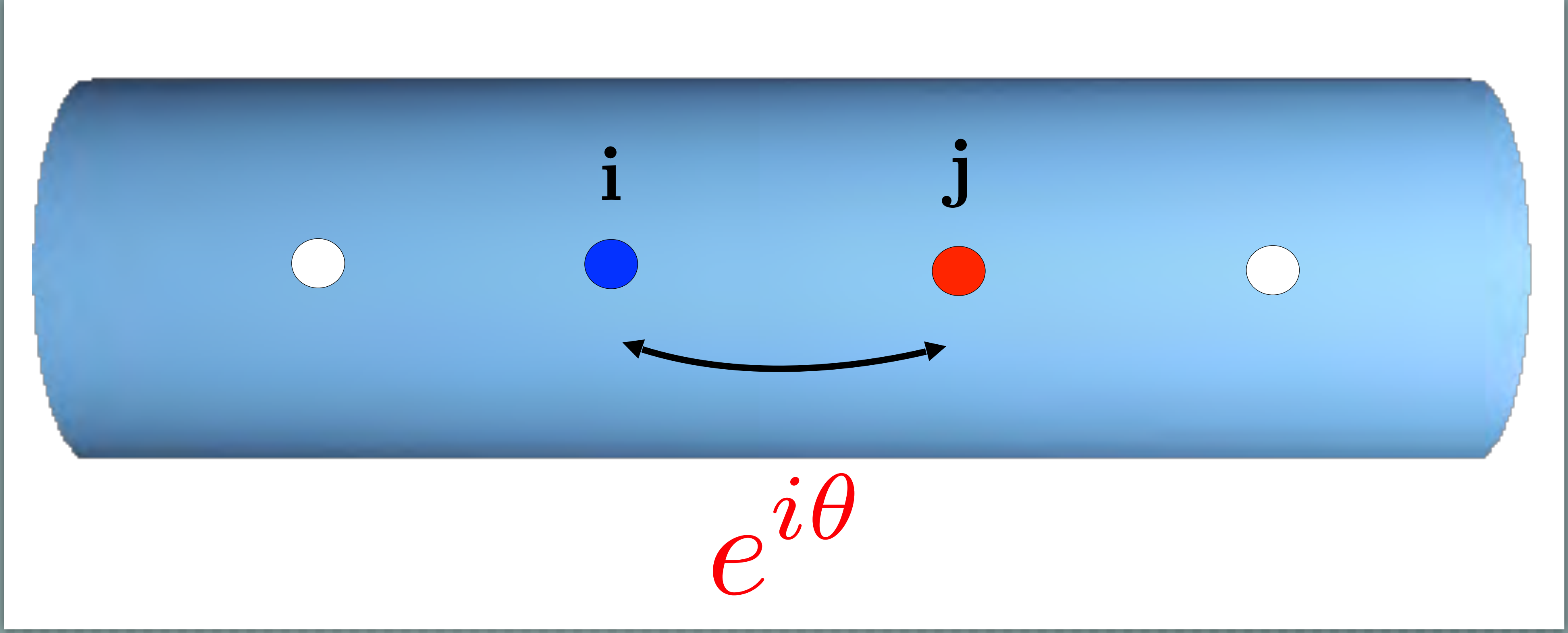
$$K^\theta_{\mathbf{j}} = \exp[i\frac{\theta}{\pi}\sum_{\mathbf{l}}\omega(\mathbf{l},\mathbf{j})\ \bar{n}_{\mathbf{l}}] \qquad K^{1d}_{\mathbf{j}} = \exp[i\pi\sum_{\mathbf{l}<\mathbf{j}}\ \bar{n}_{\mathbf{l}}]$$

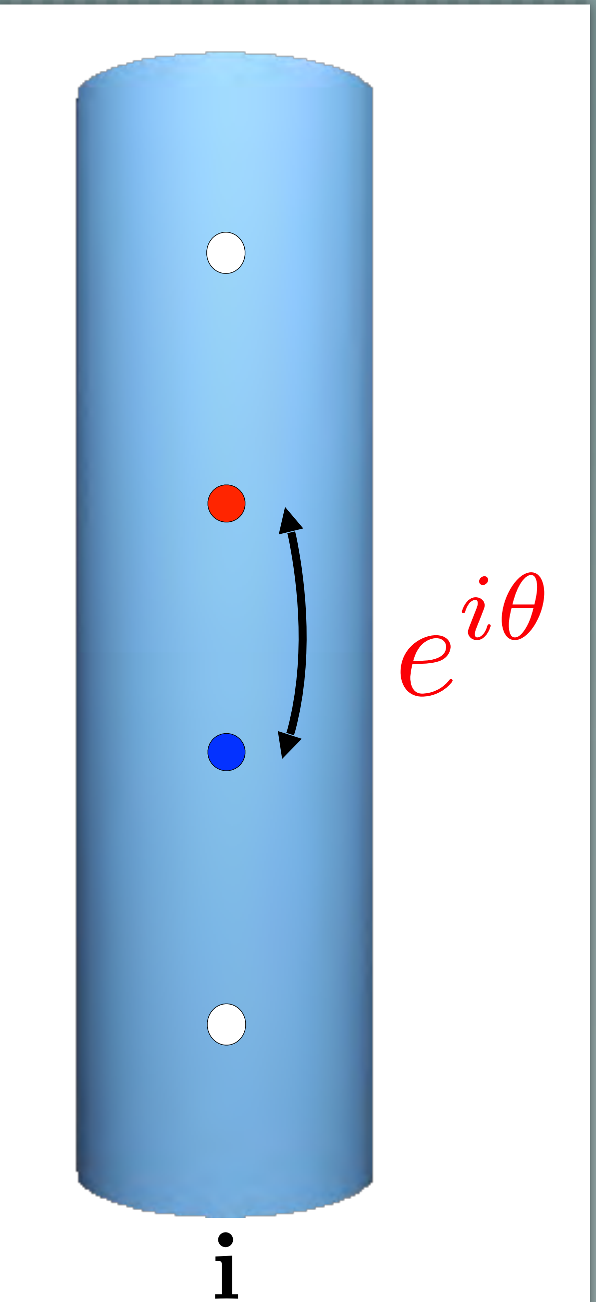
Local Transmutation:



Local Transmutation:

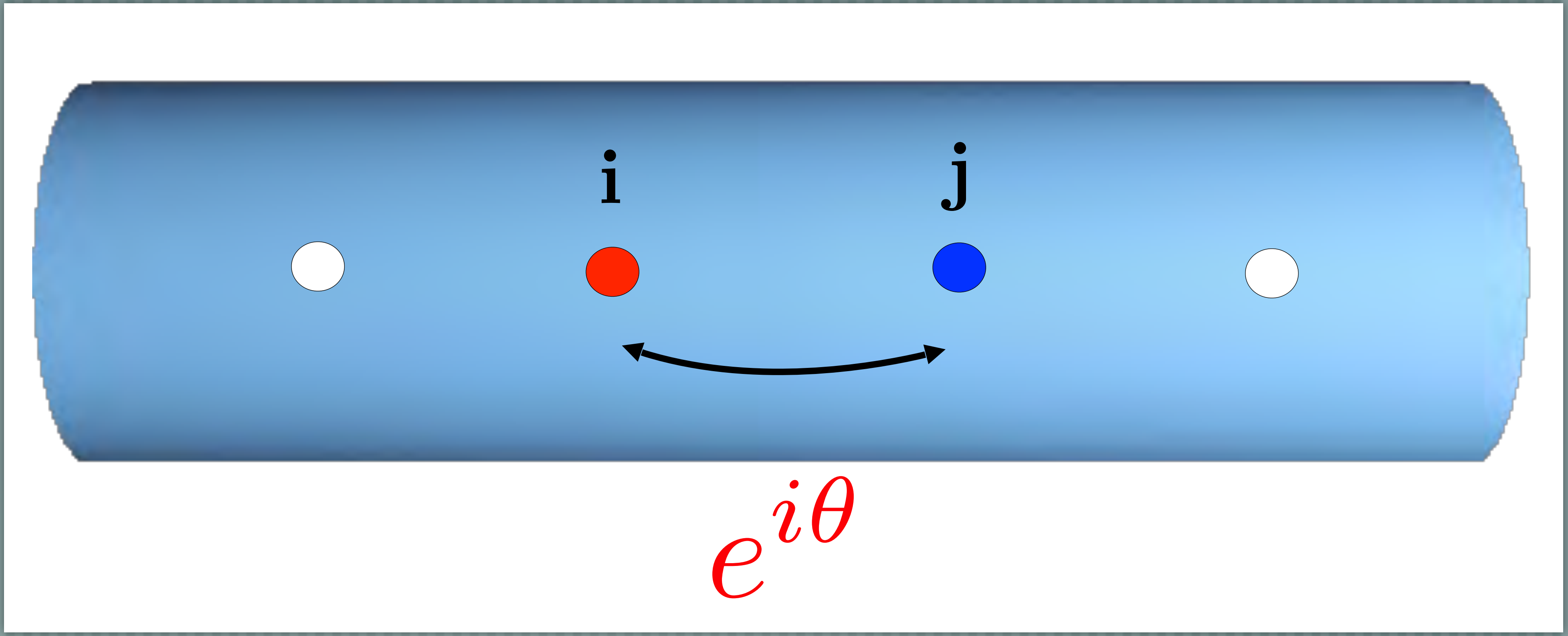




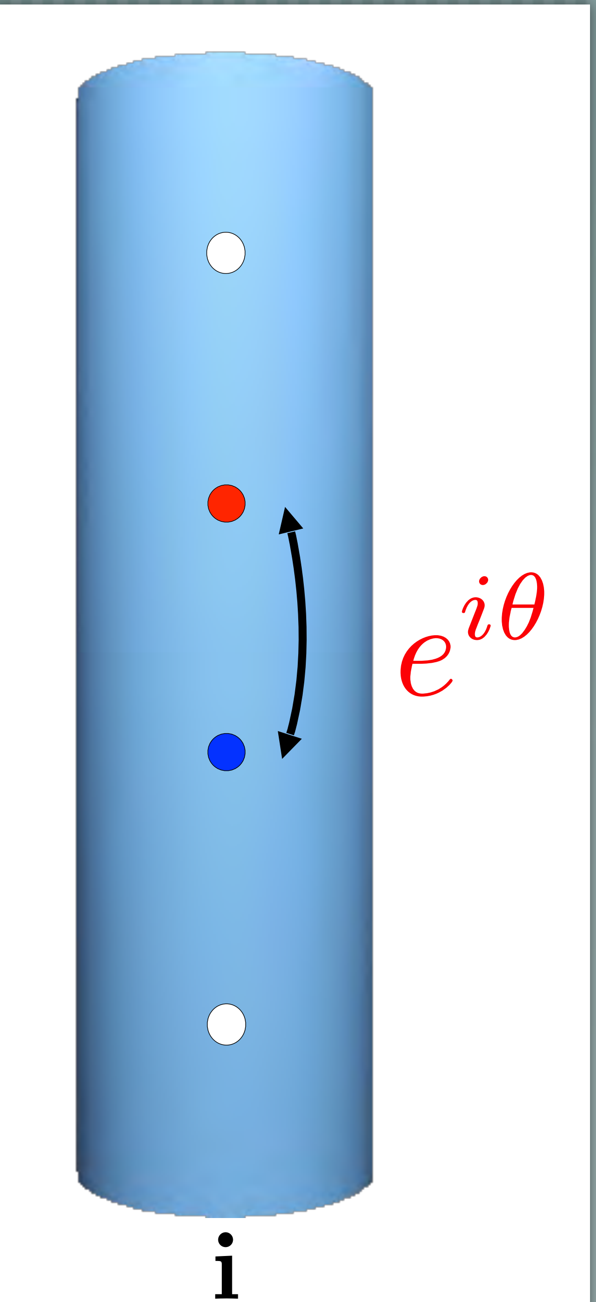


Non-local Transmutation:

Local Transmutation:



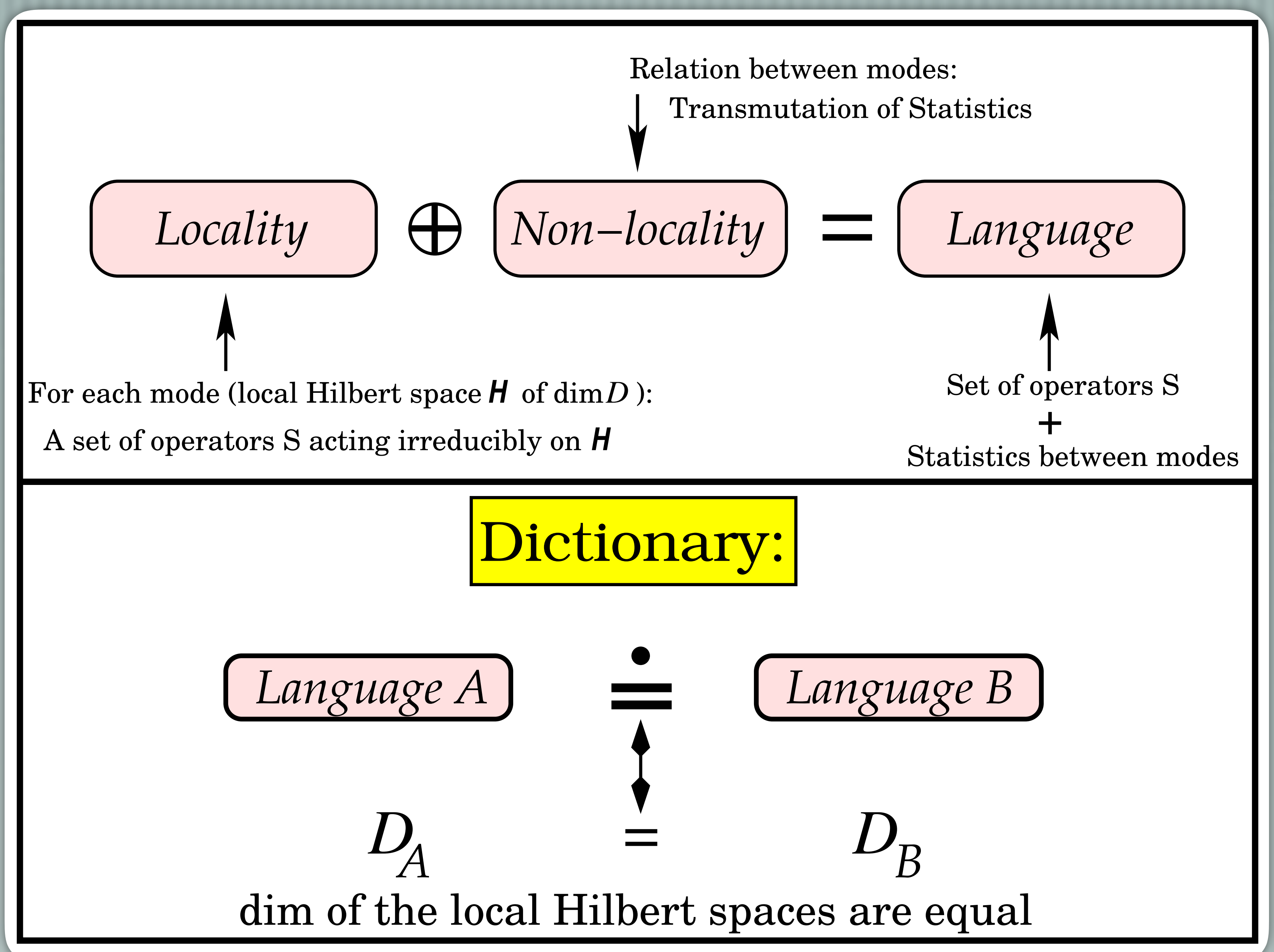
Non-local Transmutation:



Local Transmutation:

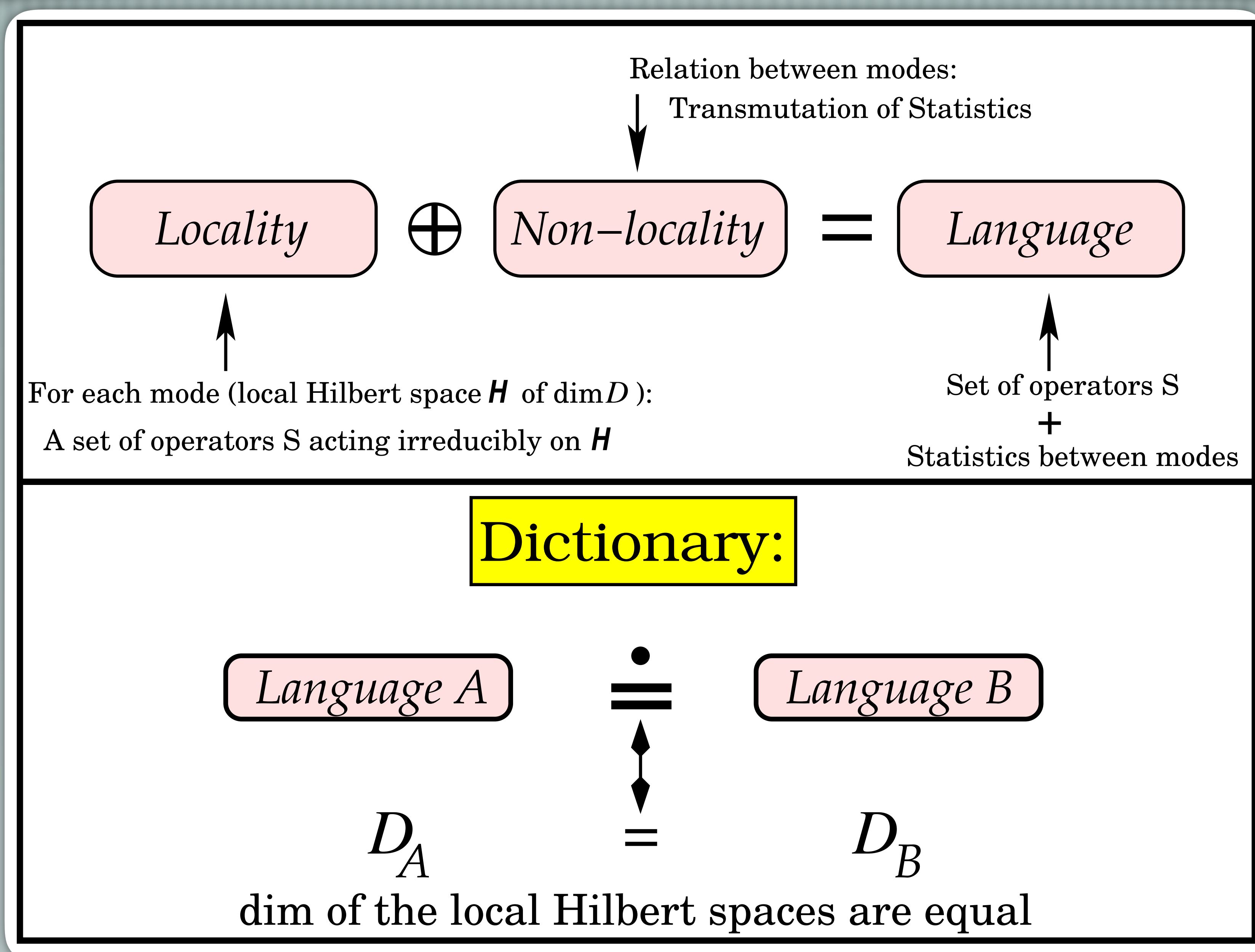
The Languages of Nature

- We have shown how to connect all possible (spin-particle-gauge) languages used in the quantum description of matter, and proved a fundamental theorem that establishes when two languages can be connected through a dictionary.



The Languages of Nature

Defining a Language amounts to defining the State Space

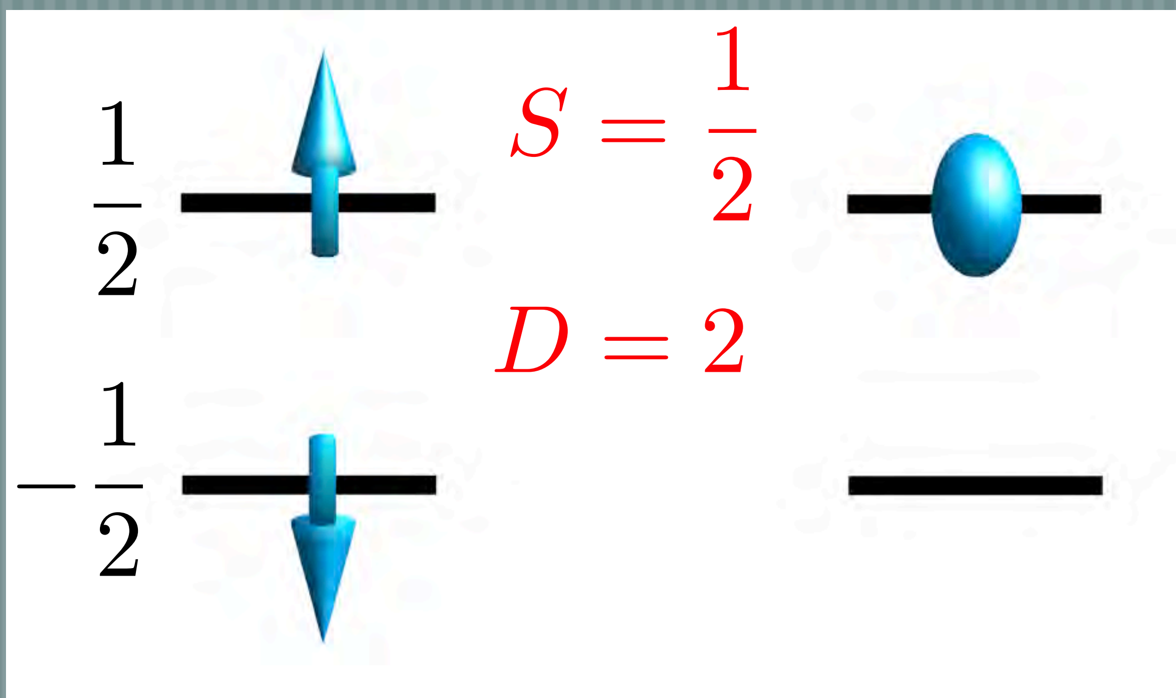


Can one connect the different Languages (spin-fermion-boson-gauge)?

Building Dictionaries

Jordan-Wigner Particles

$SU(2)$ Spin   Hard-core P

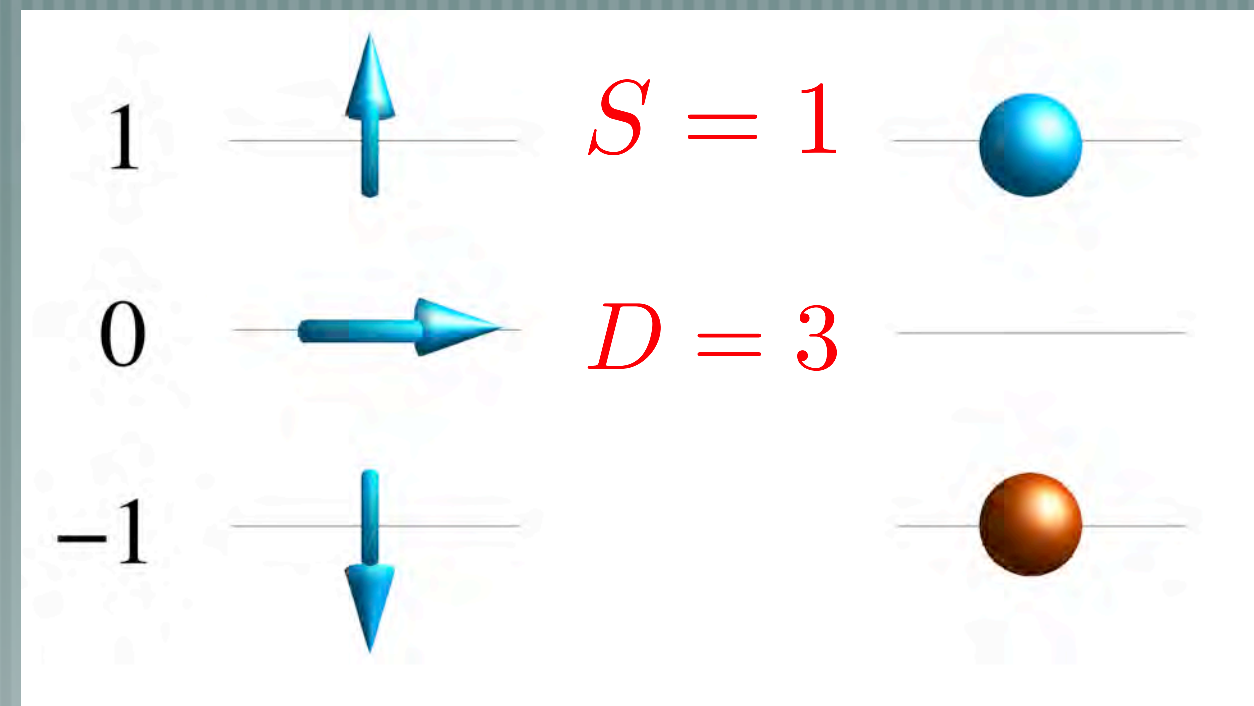


$$\tilde{b}_{\mathbf{j}}^\dagger = S_{\mathbf{j}}^+ = c_{\mathbf{j}}^\dagger \; K_{\mathbf{j}}$$

$$\tilde{b}_{\mathbf{j}} = S_{\mathbf{j}}^- = K_{\mathbf{j}}^\dagger \; c_{\mathbf{j}}$$

$$\tilde{n}_{\mathbf{j}} - \frac{1}{2} = S_{\mathbf{j}}^z = n_{\mathbf{j}} - \frac{1}{2}$$

$SU(2)$ Spin   Hard-core P



$$S_{\mathbf{j}}^+ = \sqrt{2}\,(\bar{c}_{\mathbf{j}\uparrow}^\dagger \; K_{\mathbf{j}} + K_{\mathbf{j}}^\dagger \; \bar{c}_{\mathbf{j}\downarrow})$$

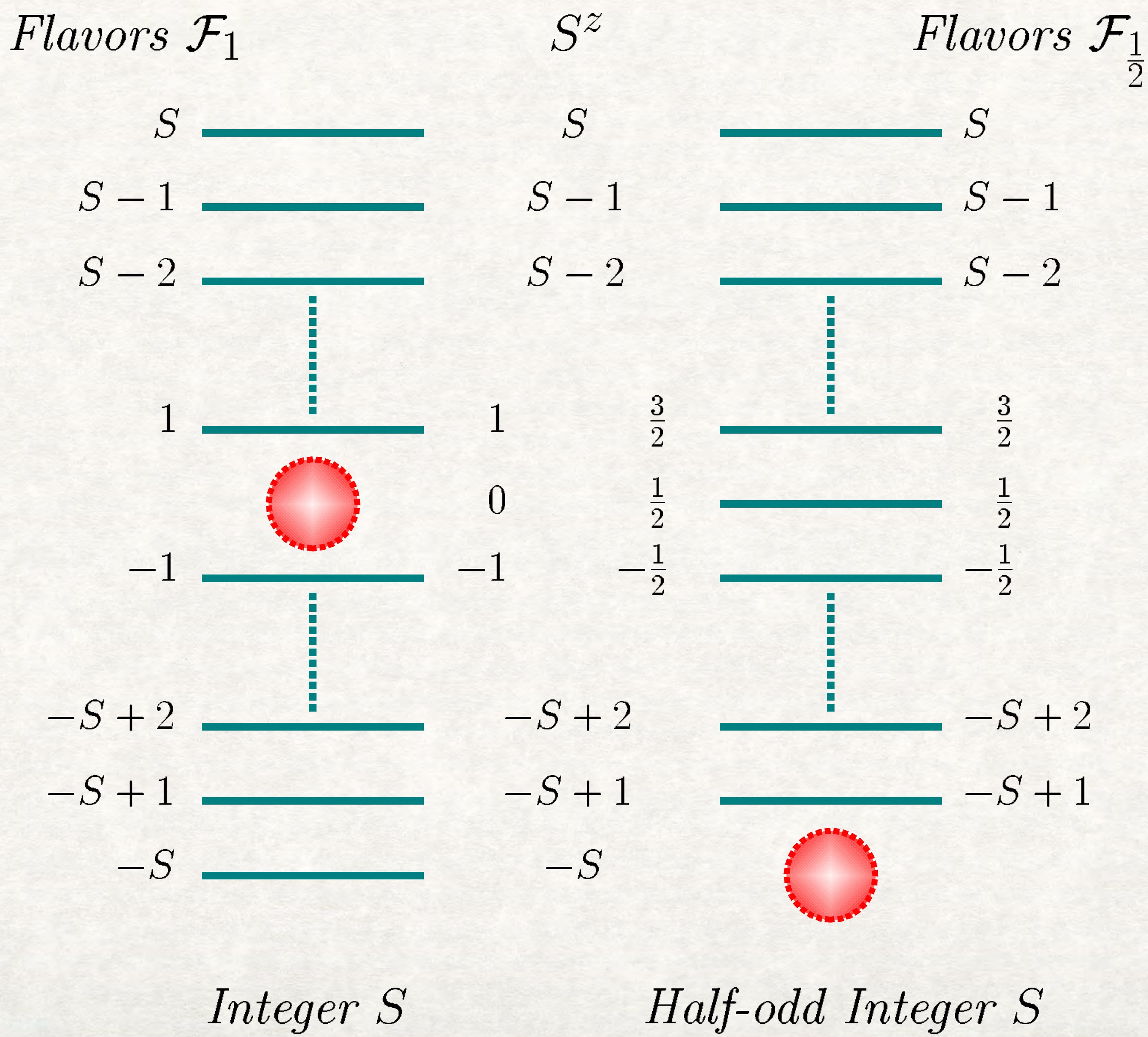
$$S_{\mathbf{j}}^- = \sqrt{2}\,(K_{\mathbf{j}}^\dagger \; \bar{c}_{\mathbf{j}\uparrow} + \bar{c}_{\mathbf{j}\downarrow}^\dagger \; K_{\mathbf{j}})$$

$$S_{\mathbf{j}}^z = \bar{n}_{\mathbf{j}\uparrow} - \bar{n}_{\mathbf{j}\downarrow}$$

$$\bar{c}_{\mathbf{j}\sigma}^\dagger = c_{\mathbf{j}\sigma}^\dagger (1 - \hat{n}_{\mathbf{j}\bar{\sigma}}) \; , \;\; \sigma = \uparrow, \downarrow$$

Generalized Jordan-Wigner Transformations

Spin-Flavor Equivalence



Some Applications Exploiting The Dictionaries

Connecting seemingly unrelated phenomena: Haldane-gap

- Haldane conjecture: Half-odd integer spin chains have a qualitative different excitation spectrum than integer spin chains

- $$H_{\mathsf{xxz}}^{S=1} = \sum_j S_j^z S_{j+1}^z + \Delta \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right)$$

$$\Updownarrow$$

$$H_{\mathsf{xxz}}^{S=1} = \sum_j (\bar{n}_{j\uparrow} - \bar{n}_{j\downarrow})(\bar{n}_{j+1\uparrow} - \bar{n}_{j+1\downarrow}) + \Delta \sum_{j\sigma} \left(\bar{c}^\dagger_{j\sigma} \bar{c}_{j+1\sigma} + \bar{c}^\dagger_{j\sigma} \bar{c}^\dagger_{j+1\bar{\sigma}} + \text{H.c.} \right)$$

- $H_{\mathsf{xxz}}^{S=1}$ is a t-J_z model + SC, where $t = -\Delta$ and $J_z = 4$. *For $8|t| > J_z$, the charge spectrum of the t-J_z chain is gapless*

- In the particle language, the Haldane gap is a superconducting gap

Ferromagnetism and Bose-Einstein Condensation

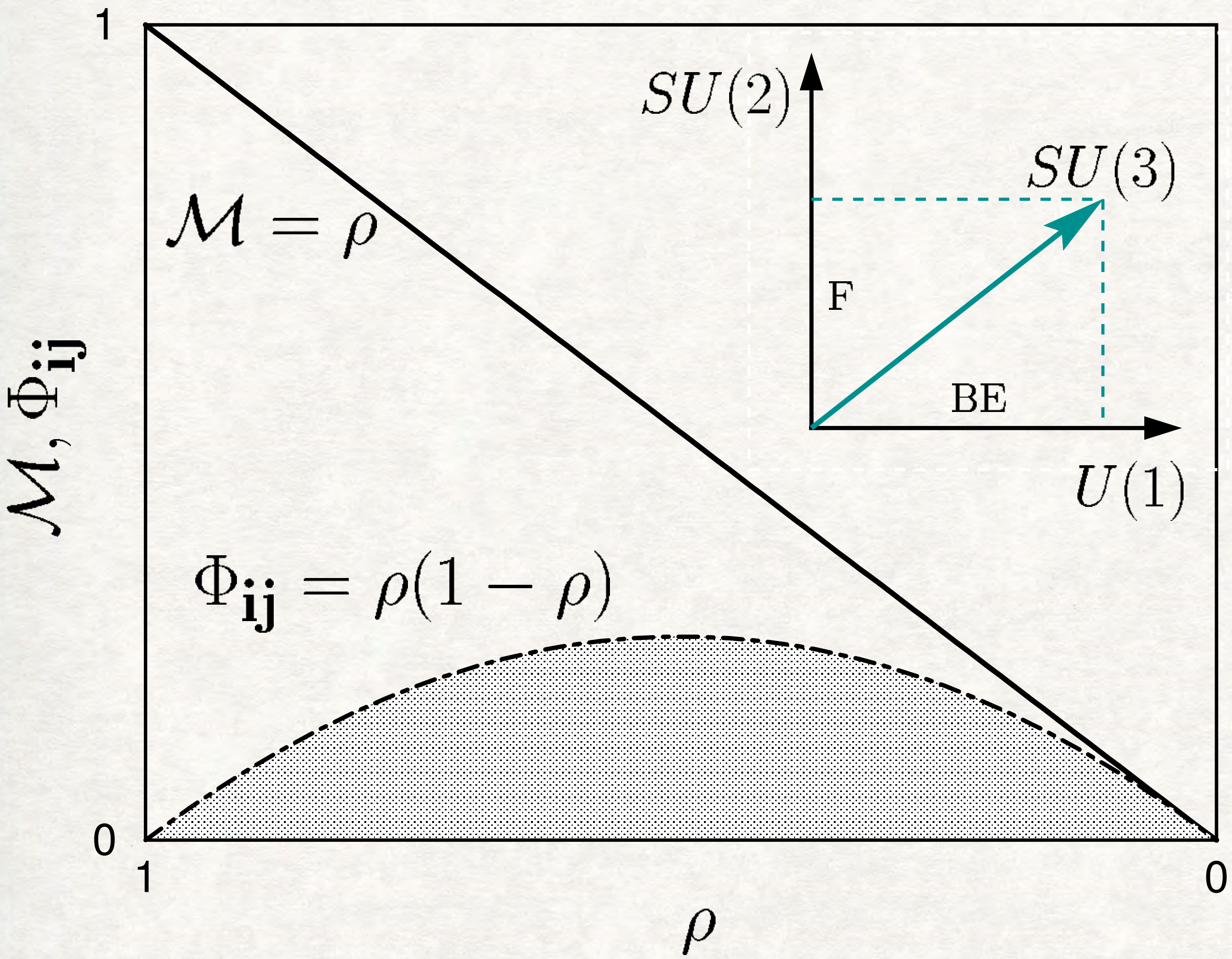
$$(T=0), \quad J>0$$

$$H_\theta(J) = J\sqrt{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left[\cos\theta\, \mathbf{S}_\mathbf{i}\cdot\mathbf{S}_\mathbf{j} + \sin\theta\,(\mathbf{S}_\mathbf{i}\cdot\mathbf{S}_\mathbf{j})^2\right]$$

$$\text{(SPT)} \Downarrow \qquad \theta = \frac{5\pi}{4}$$

$$\begin{aligned} H_{\frac{5\pi}{4}}(J) &= -J\sum_{\langle \mathbf{i},\mathbf{j}\rangle,\sigma}\left(\bar{b}^\dagger_{\mathbf{i}\sigma}\bar{b}_{\mathbf{j}\sigma} + \text{H.c.}\right) - 2J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\mathbf{s}_\mathbf{i}\cdot\mathbf{s}_\mathbf{j} \\ &- 2J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left(1-\frac{\bar{n}_\mathbf{i}+\bar{n}_\mathbf{j}}{2}+\frac{3}{4}\bar{n}_\mathbf{i}\bar{n}_\mathbf{j}\right) = -2J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}P_s(\mathbf{i},\mathbf{j}) \end{aligned}$$

Broken Symmetries



$H_{\frac{5\pi}{4}}$ has a global $SU(3)$ symmetry

BS: $(\mathcal{N} = \mathcal{N}_\uparrow + \mathcal{N}_\downarrow \le N_s)$

- $SU(2)$ of spin $(s = \frac{1}{2})$
- $U(1)$ of charge

Ground State: $|\Psi_0(\mathcal{N}, S_z)\rangle = (\tilde{b}^\dagger_{\mathbf{0}\uparrow})^{\mathcal{N}_\uparrow} (\tilde{b}^\dagger_{\mathbf{0}\downarrow})^{\mathcal{N}_\downarrow} |0\rangle$

Goldstone modes: $$\begin{cases} |\Psi^h_{\mathbf{k}}(\mathcal{N}, S_z)\rangle = \tilde{b}_{\mathbf{k}\sigma} |\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasihole,} \\ |\Psi^p_{\mathbf{k}}(\mathcal{N}, S_z)\rangle = \tilde{b}^\dagger_{\mathbf{k}\sigma} |\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasiparticle} \end{cases}$$

Hierarchical Language

$$H_\theta(J) = J\sqrt{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left[\cos\theta\,\mathbf{S_i}\cdot\mathbf{S_j} + \sin\theta\,(\mathbf{S_i}\cdot\mathbf{S_j})^2\right]$$

$\Downarrow$ ($SU(3)$ SPT)

$$H_\theta(J) = J\sqrt{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left[\cos\theta\,\mathcal{S}^{\mu\nu}(\mathbf{i})\mathcal{S}^{\nu\mu}(\mathbf{j}) + (\sin\theta - \cos\theta)\,\mathcal{S}^{\mu\nu}(\mathbf{i})\tilde{\mathcal{S}}^{\nu\mu}(\mathbf{j})\right]$$

Fundamental and Conjugate representations

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_{\mathbf{j}} & \bar{b}_{\mathbf{j}\uparrow} & \bar{b}_{\mathbf{j}\downarrow} \\ \bar{b}^{\dagger}_{\mathbf{j}\uparrow} & \bar{n}_{\mathbf{j}\uparrow} - \frac{1}{3} & \bar{b}^{\dagger}_{\mathbf{j}\uparrow}\bar{b}_{\mathbf{j}\downarrow} \\ \bar{b}^{\dagger}_{\mathbf{j}\downarrow} & \bar{b}^{\dagger}_{\mathbf{j}\downarrow}\bar{b}_{\mathbf{j}\uparrow} & \bar{n}_{\mathbf{j}\downarrow} - \frac{1}{3} \end{pmatrix}$$

"Quark"

$$\tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_{\mathbf{j}} & -\bar{b}^{\dagger}_{\mathbf{j}\downarrow} & -\bar{b}^{\dagger}_{\mathbf{j}\uparrow} \\ -\bar{b}_{\mathbf{j}\downarrow} & \bar{n}_{\mathbf{j}\downarrow} - \frac{1}{3} & \bar{b}^{\dagger}_{\mathbf{j}\uparrow}\bar{b}_{\mathbf{j}\downarrow} \\ -\bar{b}_{\mathbf{j}\uparrow} & \bar{b}^{\dagger}_{\mathbf{j}\downarrow}\bar{b}_{\mathbf{j}\uparrow} & \bar{n}_{\mathbf{j}\uparrow} - \frac{1}{3} \end{pmatrix}$$

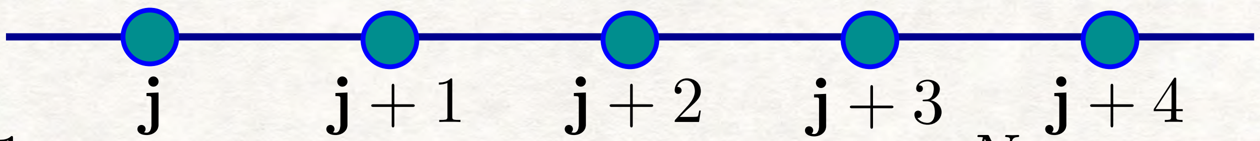
"Anti – Quark"

$$[\mathcal{S}^{\mu\mu'}(\mathbf{j}), \mathcal{S}^{\nu\nu'}(\mathbf{j})] = \delta_{\mu'\nu}\mathcal{S}^{\mu\nu'}(\mathbf{j}) - \delta_{\mu\nu'}\mathcal{S}^{\nu\mu'}(\mathbf{j})$$

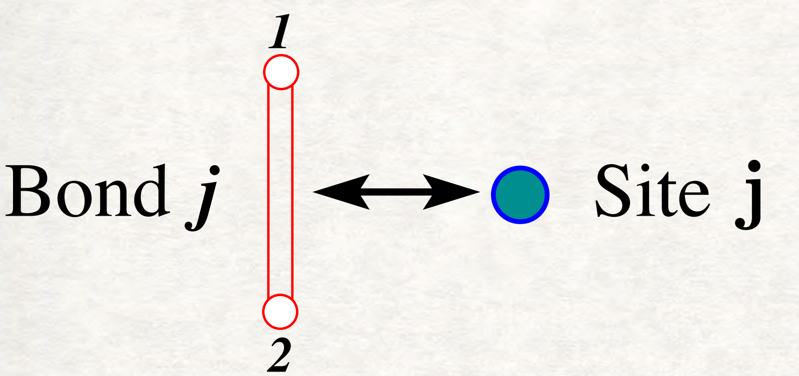
Global $SU(3)$ Order Parameters

θ	Global $SU(3)$ OP	OP 1	OP 2
$5\pi/4$ (FM-UN)	$\mathcal{S} = \sum_{\mathbf{j}} \mathcal{S}(\mathbf{j})$	$\mathbf{M} = \sum_{\mathbf{j}} \mathbf{S}_{\mathbf{j}}$	$\mathbf{N} = \sum_{\mathbf{j}} \mathbf{N}_{\mathbf{j}}$
$\pi/4$ (AF-SN)	$\mathcal{S}_{\mathsf{ST}} = \sum_{\mathbf{j}} \mathcal{S}(\mathbf{j}) e^{i\mathbf{Q}\cdot\mathbf{r}_{\mathbf{j}}}$	$\mathbf{M}_{\mathsf{ST}} = \sum_{\mathbf{j}} \mathbf{S}_{\mathbf{j}} e^{i\mathbf{Q}\cdot\mathbf{r}_{\mathbf{j}}}$	$\mathbf{N}_{\mathsf{ST}} = \sum_{\mathbf{j}} \mathbf{N}_{\mathbf{j}} e^{i\mathbf{Q}\cdot\mathbf{r}_{\mathbf{j}}}$
$3\pi/2$ (AF-UN)	$\mathcal{S}_{+} = \sum_{\mathbf{j}\in A} \mathcal{S}(\mathbf{j}) + \sum_{\mathbf{j}\in B} \tilde{\mathcal{S}}(\mathbf{j})$	$\mathbf{M}_{\mathsf{ST}} = \sum_{\mathbf{j}} \mathbf{S}_{\mathbf{j}} e^{i\mathbf{Q}\cdot\mathbf{r}_{\mathbf{j}}}$	$\mathbf{N} = \sum_{\mathbf{j}} \mathbf{N}_{\mathbf{j}}$
$\pi/2$ (FM-SN)	$\mathcal{S}_{-} = \sum_{\mathbf{j}\in A} \mathcal{S}(\mathbf{j}) - \sum_{\mathbf{j}\in B} \tilde{\mathcal{S}}(\mathbf{j})$	$\mathbf{M} = \sum_{\mathbf{j}} \mathbf{S}_{\mathbf{j}}$	$\mathbf{N}_{\mathsf{ST}} = \sum_{\mathbf{j}} \mathbf{N}_{\mathbf{j}} e^{i\mathbf{Q}\cdot\mathbf{r}_{\mathbf{j}}}$

■ Finding useful translations: The one-dimensional Hubbard model



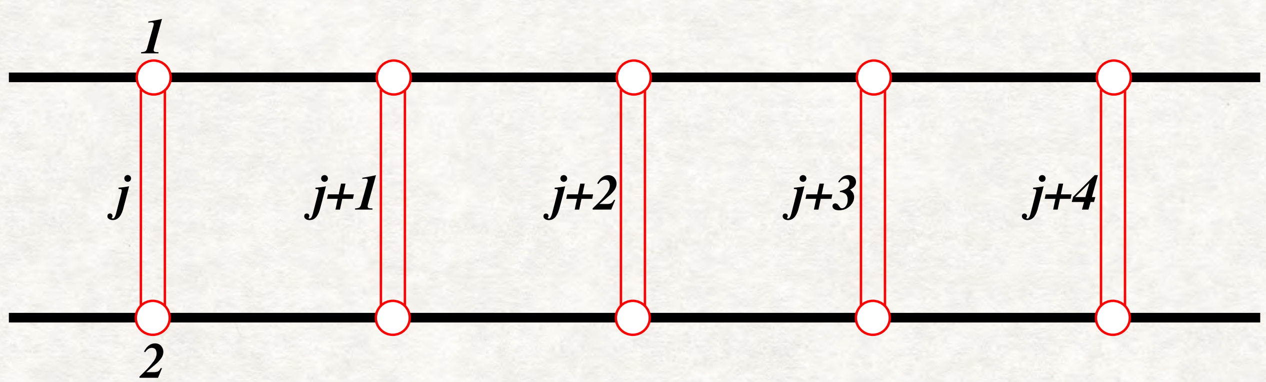
$$H^{1d}_{\text{Hubb}} = t \sum_{\mathbf{j},\sigma}^{N-1} (c^\dagger_{\mathbf{j}\sigma} c_{\mathbf{j+1}\sigma} + c^\dagger_{\mathbf{j}+1\sigma} c_{\mathbf{j}\sigma}) + U \sum_{\mathbf{j}=1}^{N} (\hat{n}_{\mathbf{j}\uparrow} - \frac{1}{2})(\hat{n}_{\mathbf{j}\downarrow} - \frac{1}{2})$$



$$\begin{cases} S^+_{\mathbf{j}1} = c^\dagger_{\mathbf{j}\uparrow} \bar{K}_{\mathbf{j}\uparrow}, \\ S^z_{\mathbf{j}1} = \hat{n}_{\mathbf{j}\uparrow} - \frac{1}{2}, \\ S^+_{\mathbf{j}2} = c^\dagger_{\mathbf{j}\downarrow} \bar{K}_{\mathbf{j}\downarrow}, \\ S^z_{\mathbf{j}2} = \hat{n}_{\mathbf{j}\downarrow} - \frac{1}{2}, \end{cases}$$

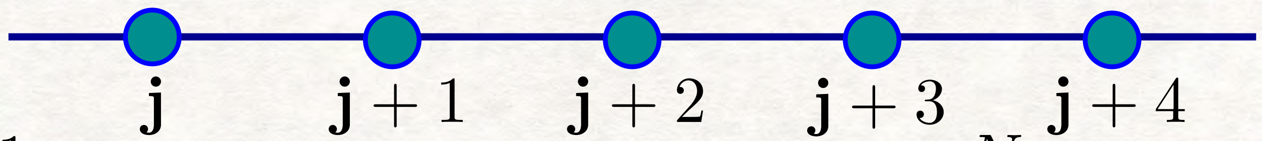
$$\bar{K}_{\mathbf{j}\uparrow} = \exp[i\pi(\sum_{\mathbf{l}} \hat{n}_{\mathbf{l}\downarrow} + \sum_{\mathbf{l}<\mathbf{j}} \hat{n}_{\mathbf{l}\uparrow})]$$

$$\bar{K}_{\mathbf{j}\downarrow} = \exp[i\pi \sum_{\mathbf{l}<\mathbf{j}} \hat{n}_{\mathbf{l}\downarrow}]$$

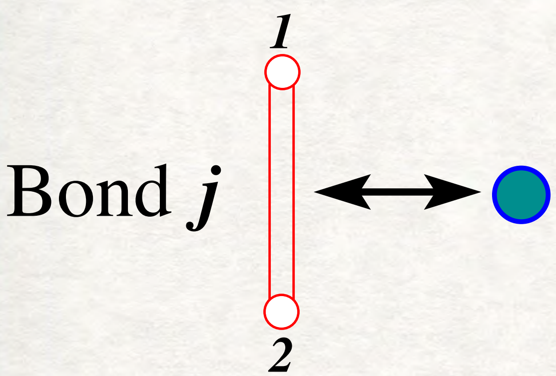


$$H^{1d}_{\text{Hubb}} = 2t \sum_{\mathbf{j},\nu}^{N-1} (S^x_{\mathbf{j}\nu} S^x_{\mathbf{j}+1\nu} + S^y_{\mathbf{j}\nu} S^y_{\mathbf{j}+1\nu}) + U \sum_{\mathbf{j}=1}^{N} S^z_{\mathbf{j}1} S^z_{\mathbf{j}2}$$

■ Finding useful translations: The one-dimensional Hubbard model



$$H^{1d}_{\mathsf{Hubb}} = t\sum_{\mathbf{j},\sigma}^{N-1}(c^\dagger_{\mathbf{j}\sigma}c_{\mathbf{j+1}\sigma} + c^\dagger_{\mathbf{j}+1\sigma}c_{\mathbf{j}\sigma}) + U\sum_{\mathbf{j}=1}^{N}(\hat{n}_{\mathbf{j}\uparrow} - \frac{1}{2})(\hat{n}_{\mathbf{j}\downarrow} - \frac{1}{2})$$



$$\begin{cases} S^+_{\mathbf{j}1} = c^\dagger_{\mathbf{j}\uparrow}\bar{K}_{\mathbf{j}\uparrow}, \\ S^z_{\mathbf{j}1} = \hat{n}_{\mathbf{j}\uparrow} - \frac{1}{2}, \\ S^+_{\mathbf{j}2} = c^\dagger_{\mathbf{j}\downarrow}\bar{K}_{\mathbf{j}\downarrow}, \\ S^z_{\mathbf{j}2} = \hat{n}_{\mathbf{j}\downarrow} - \frac{1}{2}, \end{cases}$$

$$\bar{K}_{\mathbf{j}\uparrow} = \exp[i\pi(\sum_{\mathbf{l}}\hat{n}_{\mathbf{l}\downarrow} + \sum_{\mathbf{l}<\mathbf{j}}\hat{n}_{\mathbf{l}\uparrow})]$$

$$\bar{K}_{\mathbf{j}\downarrow} = \exp[i\pi\sum_{\mathbf{l}<\mathbf{j}}\hat{n}_{\mathbf{l}\downarrow}]$$

$$H^{1d}_{\mathsf{Hubb}} = \sum_{\mathbf{j}}[J_{\mu\nu}\mathcal{S}^{\mu\nu}(\mathbf{j})\mathcal{S}^{\nu\mu}(\mathbf{j}+1) + J'_{\mu\nu}\mathcal{S}^{\mu\nu}(\mathbf{j})\tilde{\mathcal{S}}^{\nu\mu}(\mathbf{j}+1) + \frac{U}{2}(\mathcal{S}^{00}(\mathbf{j}) + \mathcal{S}^{33}(\mathbf{j}))]$$

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \mathcal{S}^{00}(\mathbf{j}) & \tilde{b}_{\mathbf{j}\uparrow}(1-\tilde{n}_{\mathbf{j}\downarrow}) & \tilde{b}_{\mathbf{j}\downarrow}(1-\tilde{n}_{\mathbf{j}\uparrow}) & \tilde{b}_{\mathbf{j}\downarrow}\tilde{b}_{\mathbf{j}\uparrow} \\ (1-\tilde{n}_{\mathbf{j}\downarrow})\tilde{b}^\dagger_{\mathbf{j}\uparrow} & \mathcal{S}^{11}(\mathbf{j}) & \tilde{b}^\dagger_{\mathbf{j}\uparrow}\tilde{b}_{\mathbf{j}\downarrow} & \tilde{n}_{\mathbf{j}\uparrow}\tilde{b}_{\mathbf{j}\downarrow} \\ (1-\tilde{n}_{\mathbf{j}\uparrow})\tilde{b}^\dagger_{\mathbf{j}\downarrow} & \tilde{b}^\dagger_{\mathbf{j}\downarrow}\tilde{b}_{\mathbf{j}\uparrow} & \mathcal{S}^{22}(\mathbf{j}) & \tilde{n}_{\mathbf{j}\downarrow}\tilde{b}_{\mathbf{j}\uparrow} \\ \tilde{b}^\dagger_{\mathbf{j}\uparrow}\tilde{b}^\dagger_{\mathbf{j}\downarrow} & \tilde{b}^\dagger_{\mathbf{j}\downarrow}\tilde{n}_{\mathbf{j}\uparrow} & \tilde{b}^\dagger_{\mathbf{j}\uparrow}\tilde{n}_{\mathbf{j}\downarrow} & \mathcal{S}^{33}(\mathbf{j}) \end{pmatrix}, \quad \tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} -\tilde{\mathcal{S}}^{00}(\mathbf{j}) & -\tilde{b}_{\mathbf{j}\uparrow}\tilde{n}_{\mathbf{j}\downarrow} & -\tilde{b}_{\mathbf{j}\downarrow}\tilde{n}_{\mathbf{j}\uparrow} & -\tilde{b}_{\mathbf{j}\downarrow}\tilde{b}_{\mathbf{j}\uparrow} \\ -\tilde{n}_{\mathbf{j}\downarrow}\tilde{b}^\dagger_{\mathbf{j}\uparrow} & -\tilde{\mathcal{S}}^{11}(\mathbf{j}) & -\tilde{b}^\dagger_{\mathbf{j}\uparrow}\tilde{b}_{\mathbf{j}\downarrow} & (\tilde{n}_{\mathbf{j}\uparrow}-1)\tilde{b}_{\mathbf{j}\downarrow} \\ -\tilde{n}_{\mathbf{j}\uparrow}\tilde{b}^\dagger_{\mathbf{j}\downarrow} & -\tilde{b}^\dagger_{\mathbf{j}\downarrow}\tilde{b}_{\mathbf{j}\uparrow} & -\tilde{\mathcal{S}}^{22}(\mathbf{j}) & (\tilde{n}_{\mathbf{j}\downarrow}-1)\tilde{b}_{\mathbf{j}\uparrow} \\ -\tilde{b}^\dagger_{\mathbf{j}\uparrow}\tilde{b}^\dagger_{\mathbf{j}\downarrow} & \tilde{b}^\dagger_{\mathbf{j}\downarrow}(\tilde{n}_{\mathbf{j}\uparrow}-1) & \tilde{b}^\dagger_{\mathbf{j}\uparrow}(\tilde{n}_{\mathbf{j}\downarrow}-1) & -\tilde{\mathcal{S}}^{33}(\mathbf{j}) \end{pmatrix},$$

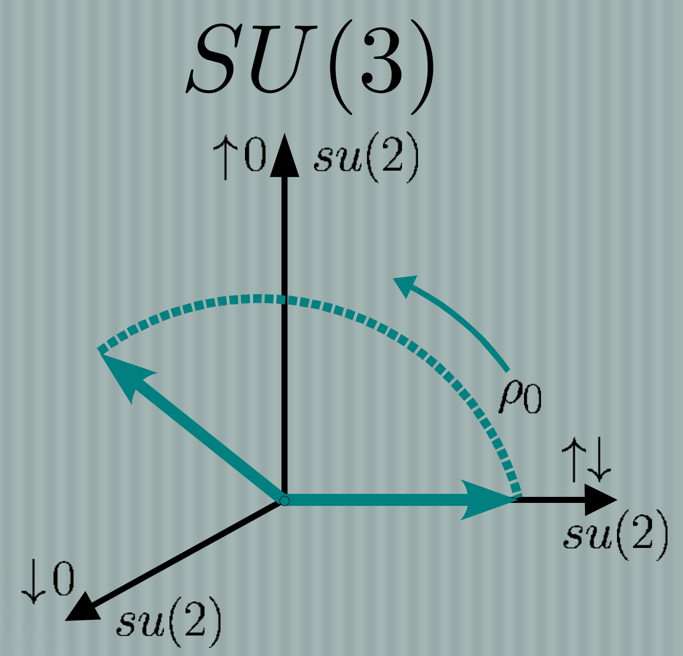
Equivalence between Spin-Particle Models

Concept of Universality (or Equivalence)

d	Model A	Model B	M - S
1	Isotropic XY (S=1/2)	t	C - E
1	Anisotropic XY (S=1/2)	t-Δ	C - E
1	XXZ (S=1/2)	t-V	C - E (BA)
1	XXZ (S=1/2)	t-J_z	P - E (BA)
1	XYZ (S=1/2)	t-Δ-V	C - E (BA)
1	Majumdar-Ghosh (S=1/2)	t-t'-V-V'	C - GS
1	BB $S = 1$ ($\phi = \pi/4$, LS)	t-$\bar{J}$-V-μ	C - E (BA)
1	BB $S = 1$ ($\tan\phi$=1/3, AKLT)	t-Δ-$\bar{J}$-V-μ	C - GS
1	BB $S = 1$ ($\phi = 7\pi/4$, TB)	t-Δ-$\bar{J}$-V-μ	C - E (BA)
1	BB $S = 1$ ($\phi = 3\pi/2$, K)	Δ-$\bar{J}$-V-μ	C - QE
1	S=3/2	F (S=1/2) Hubbard	C - E (BA)
2	$U(1)$ gauge magnet	F strings	C - E
2	Shastry-Sutherland (S=1/2)	t-t'-V-V'	C - GS
Any	BB $S = 1$ ($\phi = 5\pi/4$)	t-$\bar{J}$-V-μ	C - QE
Any	Anisotropic $S = 1$ Heisenberg	B (S=1/2) Hubbard	P - U

Classification of local Order Parameters

The generators of the HL exhaust all possible OPs
(HL: Hierarchical Language)

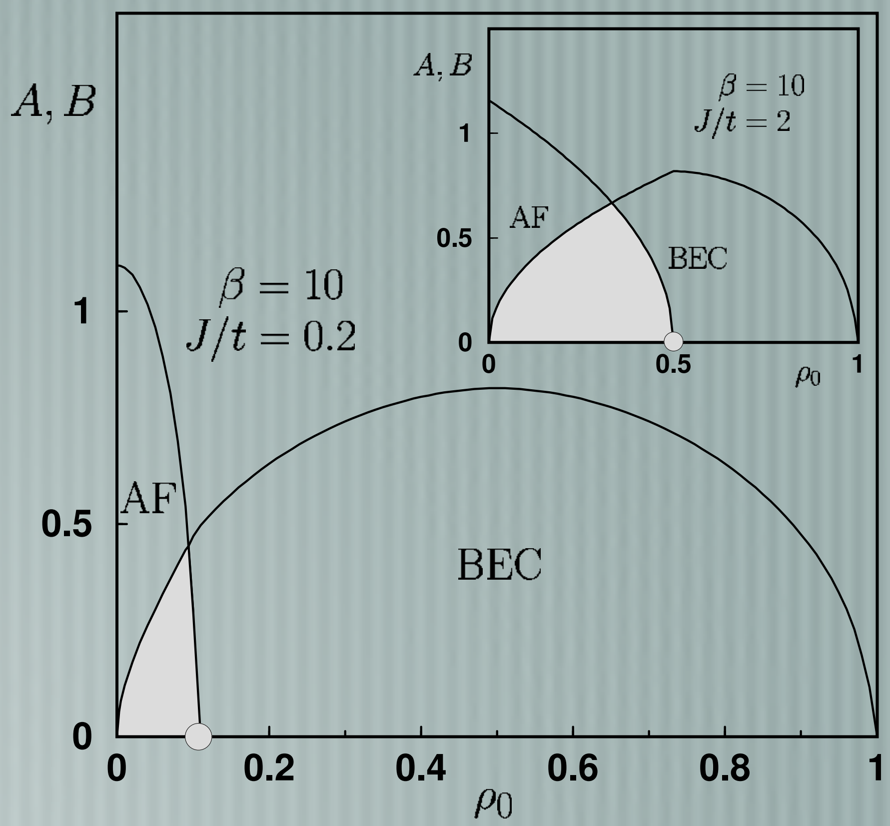


Recipe:

Identify the group of symmetries of H, G
Classify the generators of the HL according to the irreps of G
Each irrep leads to a different broken symmetry OP

Hierarchical mean-field theories

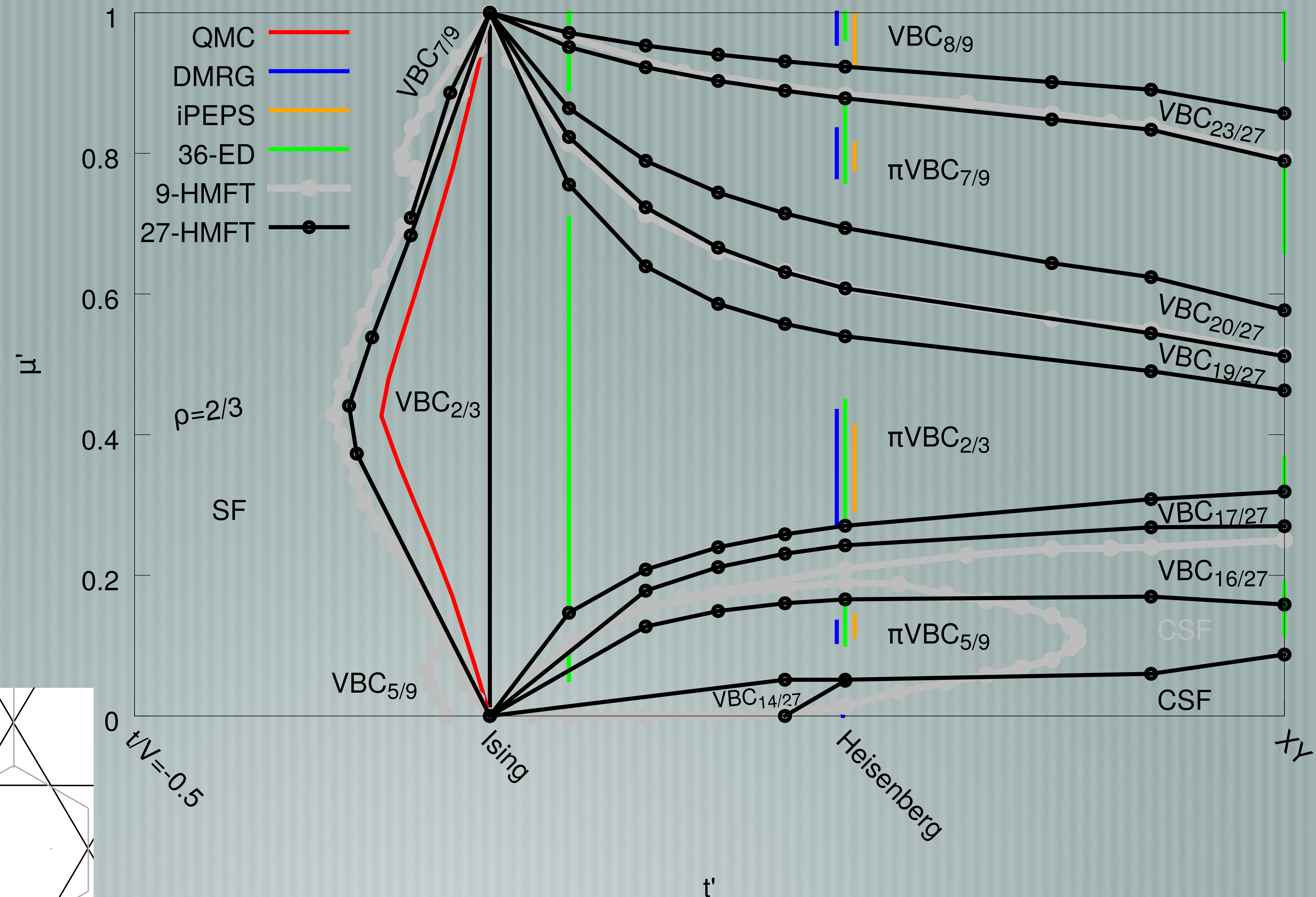
All OPs are treated on an equal footing
Help to design/engineer new states of matter

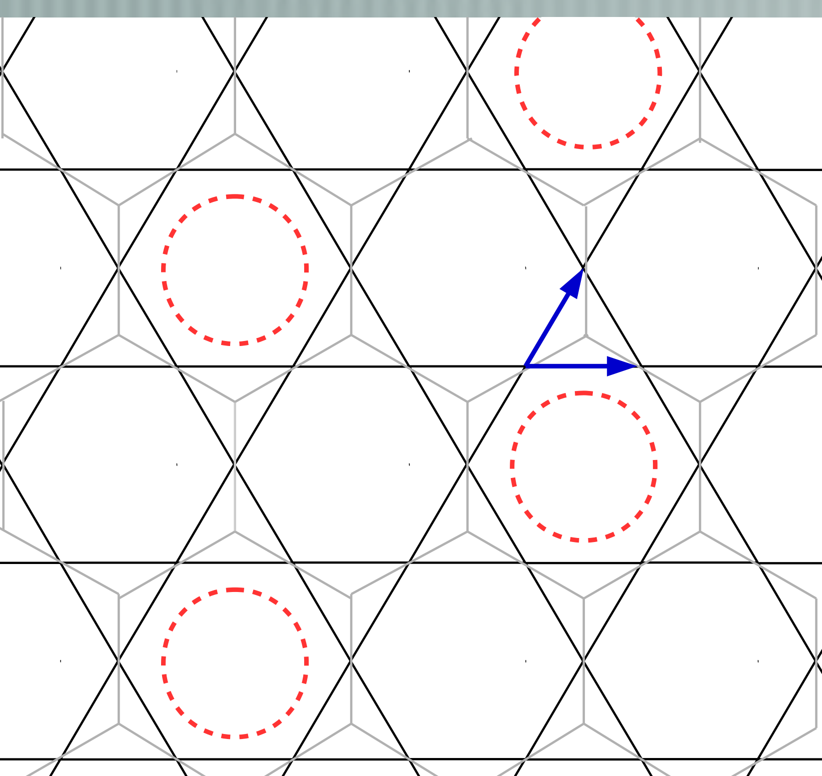


Concept of Emergent Symmetry

Helps to identify the relevant low energy degrees of freedom
(bricks to build new phases)

Hierarchical Mean-field of the Heisenberg model in Kagome





Simulation of Physical Phenomena

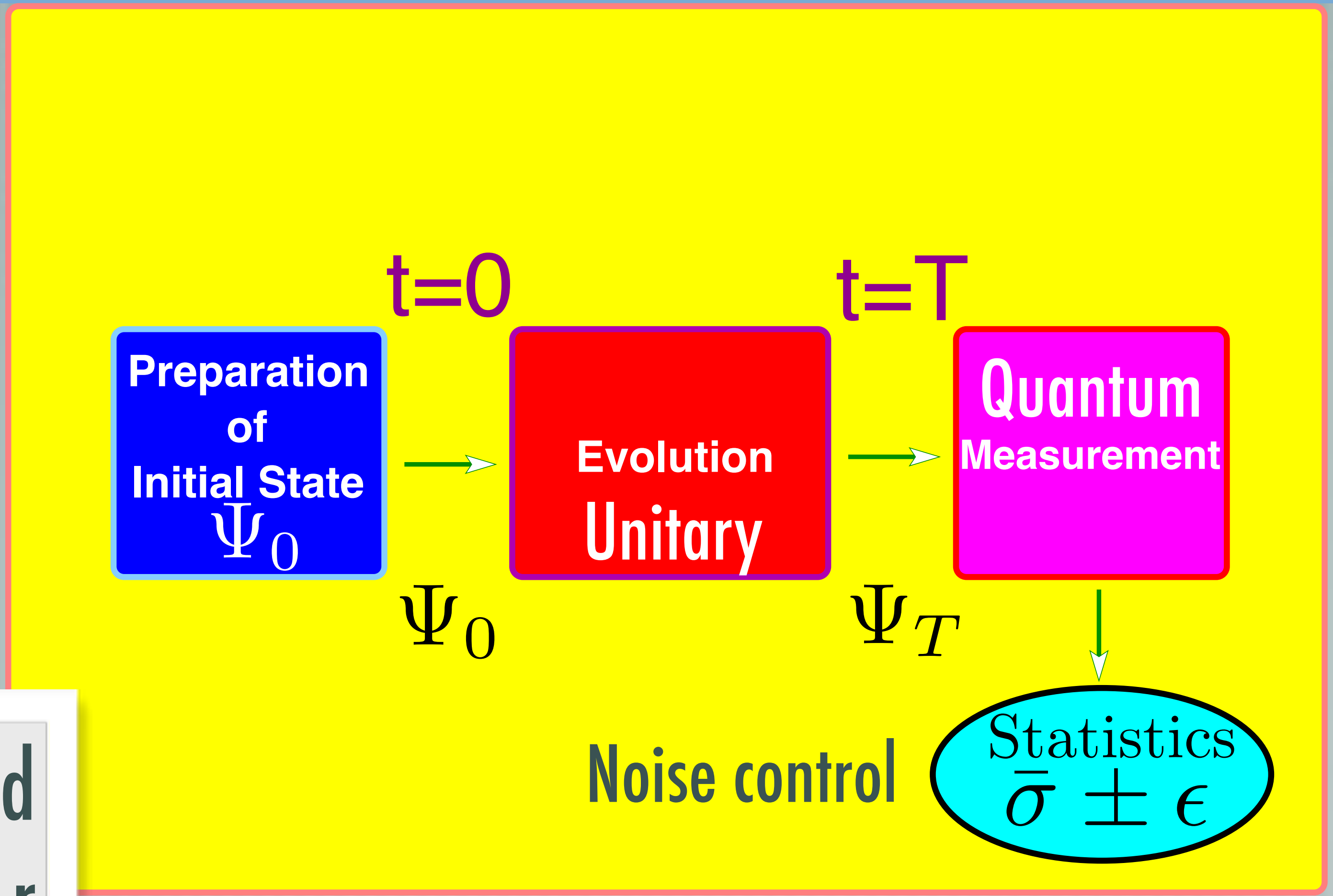
Dictionaries: Fermions and Bosons to Qubits

Equivalence between Models of Computation

Fermion Computation via the Standard (Qubit) Model

If it is possible to efficiently simulate the fermion model by the standard model then these two models of computation are equivalent

Prove that every step can be done with the same complexity



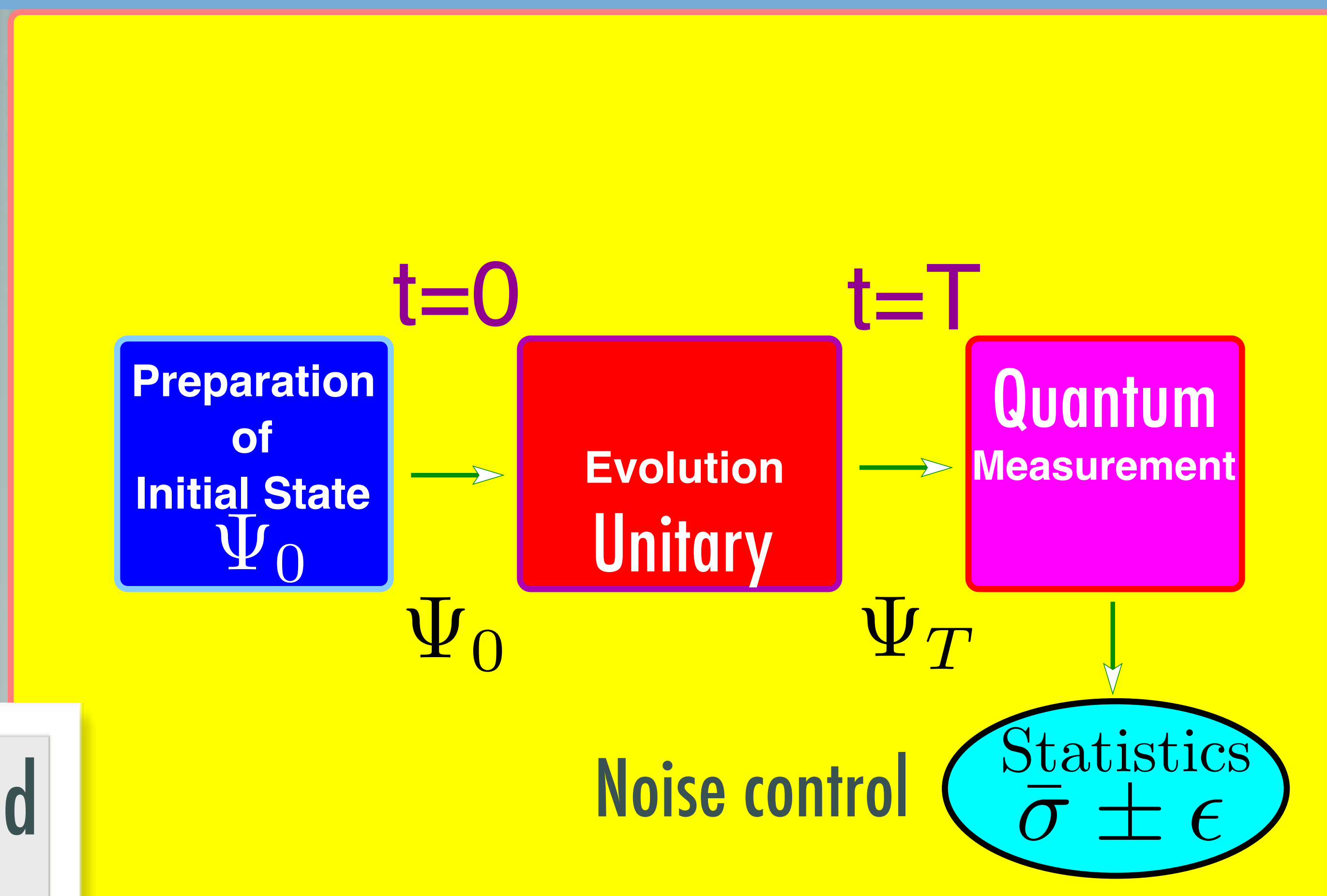
The two models are equivalent

(Tool: Generalized Jordan-Wigner mappings)

Eq on

Dynamical Correlation Functions	→	Exponential Speed-up
Shor's Factoring	→	Superpolynomial $(e^{\sqrt{n}})$ Speed-up
Grover's Search	→	Quadratic Speed-up

Prove that every step can be done with the same complexity



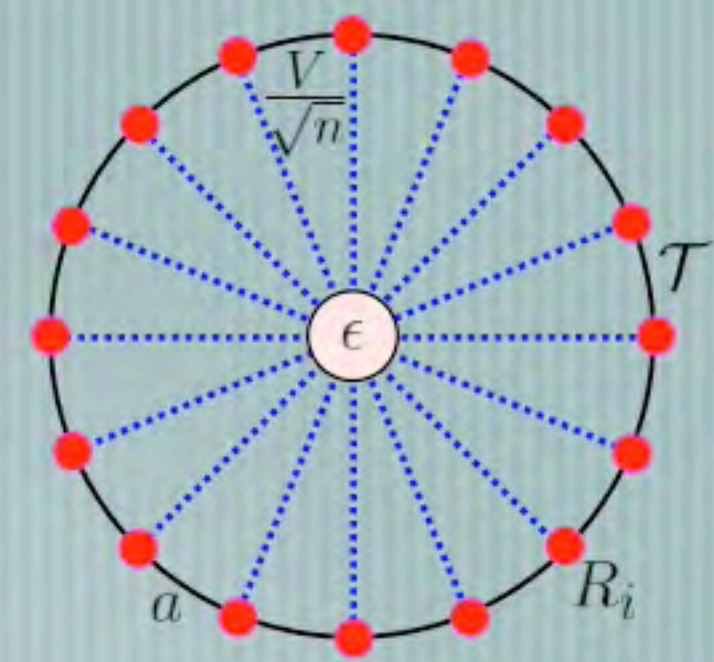
The two models are equivalent

(Tool: Generalized Jordan-Wigner mappings)

Example: Resonant Impurity Scattering

Physical System: $L = na\ ,\ R_i = ia\ ,\ (c^\dagger_{n+1} = c^\dagger_1)$

$$H = -\mathcal{T}\sum_{i=1}^{n}(c^\dagger_i c_{i+1} + c^\dagger_{i+1}c_i) + \epsilon\ b^\dagger b + \frac{V}{\sqrt{n}}\sum_{i=1}^{n}(c^\dagger_i b + b^\dagger c_i)$$



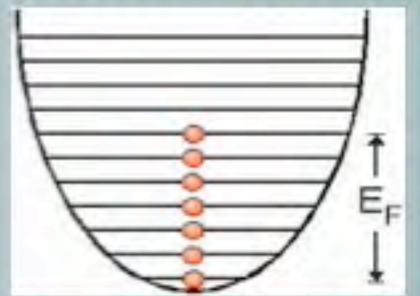
Phenomenon to study: Probability to stay in $|\Psi(0)\rangle$

$$G(t) = \langle\Psi(0)|b(t)b^\dagger(0)|\Psi(0)\rangle\ ,\ b(t) = e^{iHt}\ b(0)\ e^{-iHt}$$

Initial state: Fermi sea of $N_e \leq n$ fermions

$$\mathcal{E}_{k_i} = -2\mathcal{T}\cos k_i a$$

$$|\Psi(0)\rangle = |\mathsf{FS}\rangle = \prod_{i=0}^{N_e-1} c^\dagger_{k_i}|0\rangle\ ,\ c^\dagger_{k_i} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} e^{ik_iR_j}\ c^\dagger_j$$



Quantum Algorithm to Compute $G(t)$

Spin-Fermion mapping:

$$b = \sigma_-^1 \qquad b^\dagger = \sigma_+^1$$

$$c_{k_0} = -\sigma_z^1 \sigma_-^2 \qquad c_{k_0}^\dagger = -\sigma_z^1 \sigma_+^2$$

$$\vdots \qquad \vdots$$

$$c_{k_{n-1}} = (-1)^n \sigma_z^1 \sigma_z^2 \cdots \sigma_z^n \sigma_-^{n+1} \qquad c_{k_{n-1}}^\dagger = (-1)^n \sigma_z^1 \sigma_z^2 \cdots \sigma_z^n \sigma_+^{n+1}$$

Physical property: Effective 2-Qubit problem $G(t) = \langle W_b \rangle = \langle e^{i\bar{H}t} \sigma_-^1 e^{-i\bar{H}t} \sigma_+^1 \rangle$

$$\bar{H} = \frac{\epsilon}{2}\sigma_z^1 + \frac{\mathcal{E}_{k_0}}{2}\sigma_z^2 + \frac{V}{2}(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2)$$

Translation to NMR-machine language:

$$e^{-i\bar{H}t} = U e^{-iH_{P1}t} U^\dagger \ , \ \ H_{P1} = h_- \sigma_z^1 + h_+ \sigma_z^2$$

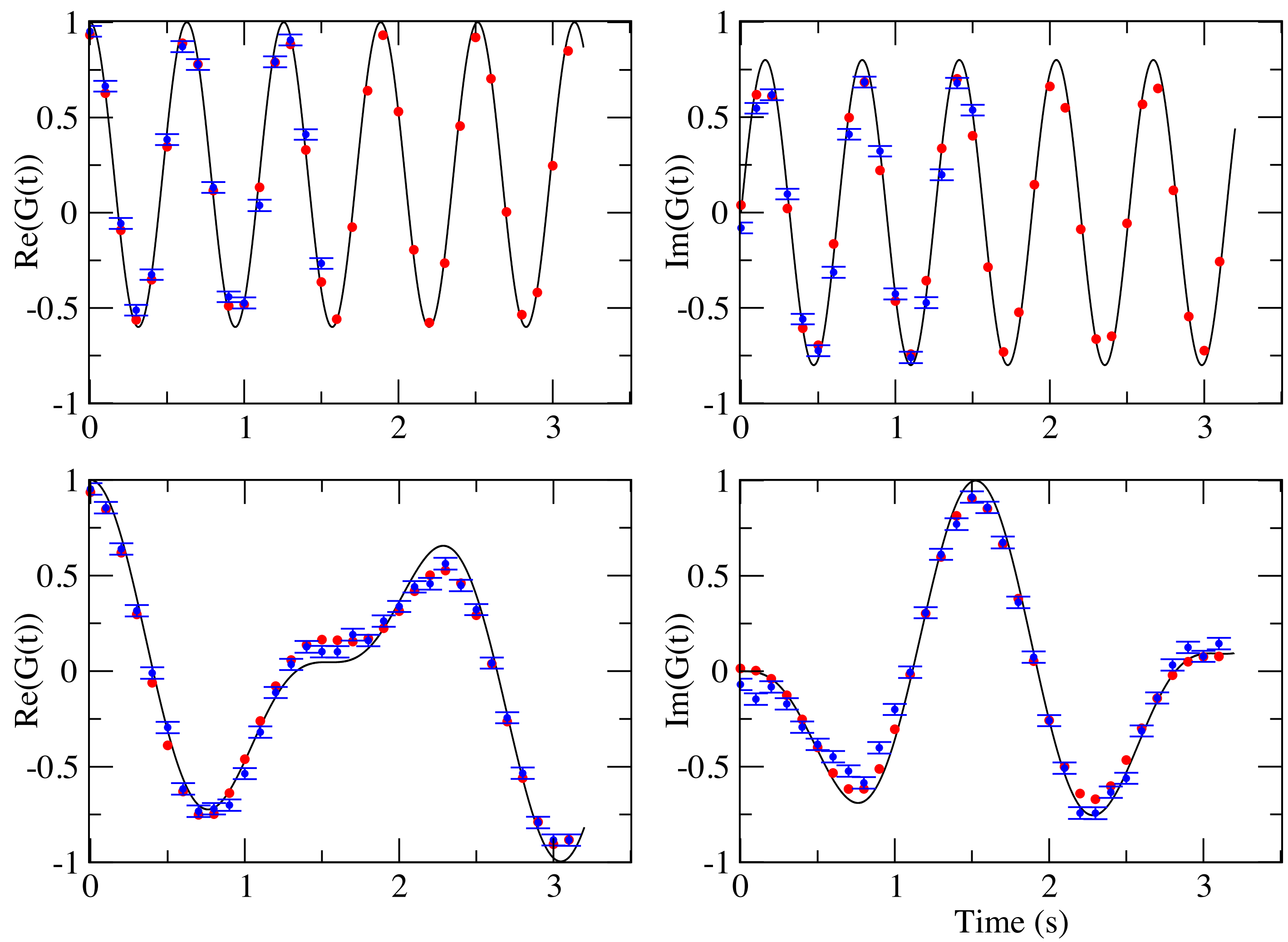
$$U = e^{i\frac{\pi}{4}\sigma_x^2} e^{-i\frac{\pi}{4}\sigma_y^1} e^{-i\frac{\theta}{2}\sigma_z^1\sigma_z^2} e^{i\frac{\pi}{4}\sigma_y^1} e^{i\frac{\pi}{4}\sigma_x^1} e^{-i\frac{\pi}{4}\sigma_x^2} e^{-i\frac{\pi}{4}\sigma_y^2} e^{i\frac{\theta}{2}\sigma_z^1\sigma_z^2} e^{-i\frac{\pi}{4}\sigma_x^1} e^{i\frac{\pi}{4}\sigma_y^2}$$

NMR-pulses

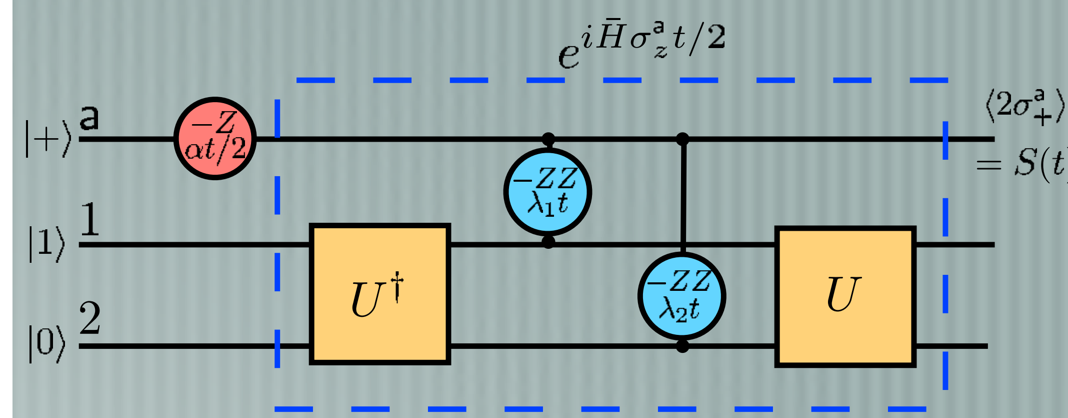
Green's Function: $G(t)$

C. Negrevergne, R. Somma, G.O., M. Knill, R. Laflamme

$\epsilon = -8 \,,\; \mathcal{E}_0 = -2 \,,\; V = 4$

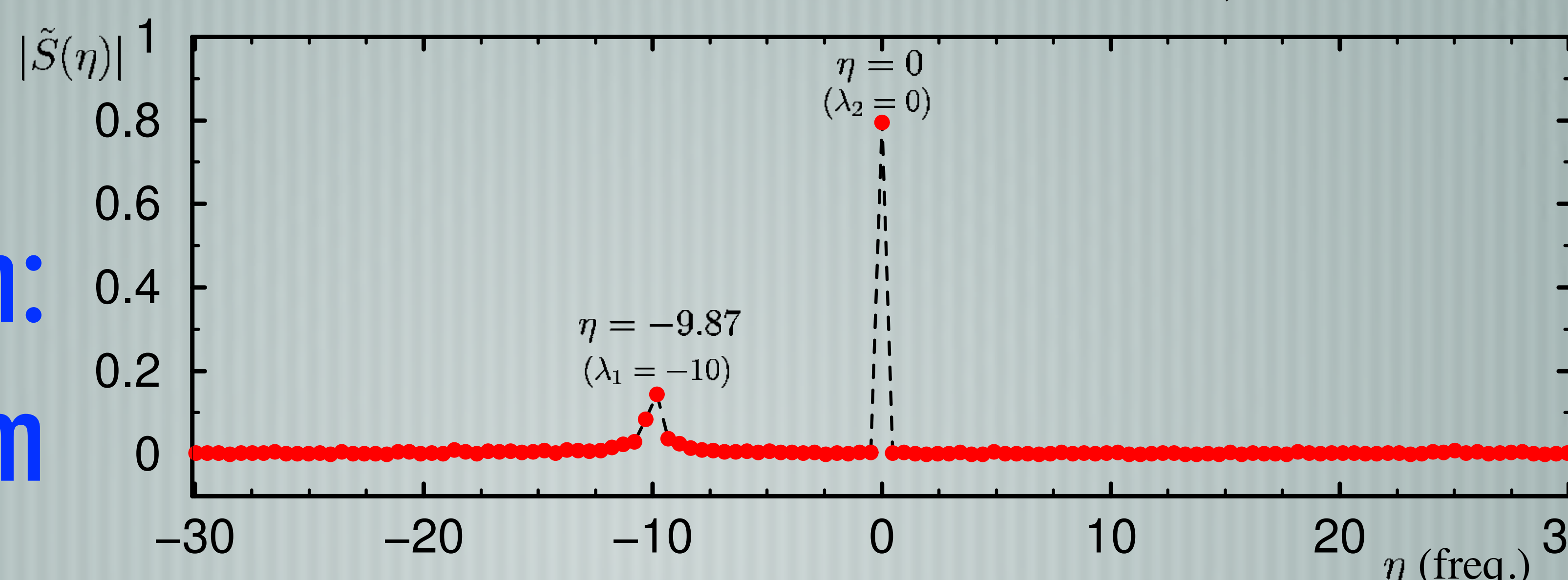


$\epsilon = 0 \,,\; \mathcal{E}_0 = -2 \,,\; V = 4$

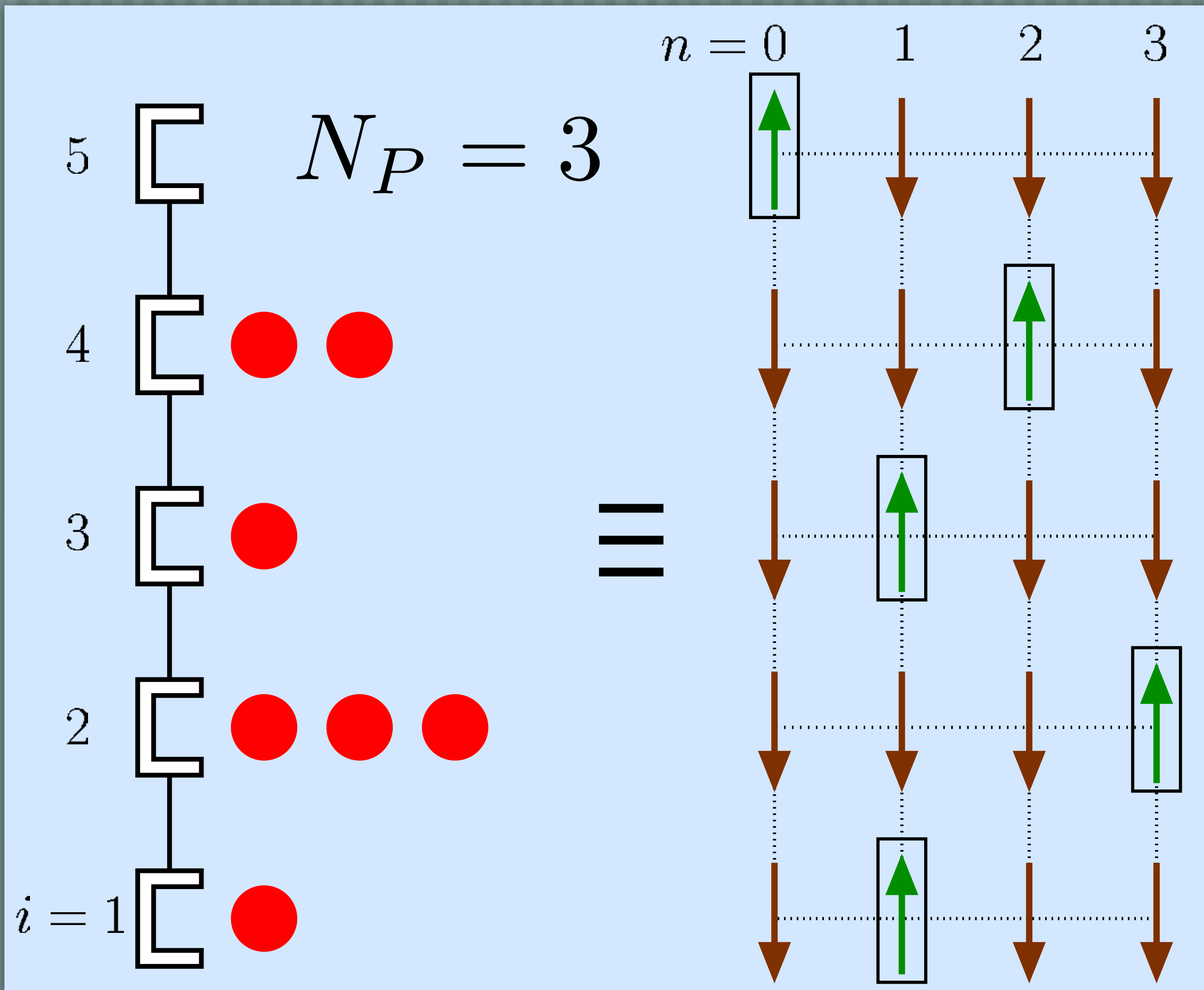


$\epsilon = -8 \,,\; \mathcal{E}_0 = -2 \,,\; V = 1/2$

Energy Spectrum: Fourier-transform



Simulating Number-conserving Canonical Bosons with Spins



$$|\phi_\alpha\rangle = \frac{1}{2\sqrt{3}} b_1^\dagger (b_2^\dagger)^3 b_3^\dagger (b_4^\dagger)^2 |\mathsf{vac}\rangle$$

$$|\phi_\alpha\rangle = |{\downarrow\uparrow\downarrow\downarrow}\rangle_1 \otimes |{\downarrow\downarrow\downarrow\uparrow}\rangle_2 \otimes |{\downarrow\uparrow\downarrow\downarrow}\rangle_3 \otimes |{\downarrow\downarrow\uparrow\downarrow}\rangle_4 \otimes |{\uparrow\downarrow\downarrow\downarrow}\rangle_5$$

$$\hat{b}^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{N_P} & 0 \end{pmatrix}$$

$$(\hat{b}^\dagger)^{N_P+1} = 0 \;\rightarrow\; \text{up to } N_P \text{ bosons}$$

$$\bar{b}_i^\dagger = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{\hat{b}^\dagger}_{i^{th}\ \text{factor}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

$$\bar{b}_i^\dagger \; |n\rangle_i = \sqrt{n+1} \; |n+1\rangle_i$$

$$\bar{b}_i^\dagger = \sum_{n=0}^{N_P-1} \sqrt{n+1} \, \sigma_-^{n,i} \sigma_+^{n+1,i}$$