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Particle

Spin
Liquid-Gas transition

Ferromagnetic transition

Hidden $\mathbb{Z}_2$ symmetry at the critical point

$\mathbb{Z}_2$ symmetry
Motivation

**WHAT?**
- Unifying framework to study interacting quantum systems:
  Dictionaries connecting the different languages of quantum mechanics
  Connection between languages and symmetries

**WHY?**
- New paradigm: coexistence and competition of “non-trivial” phases
- Determination and classification of order parameters
- Identify general symmetry principles for complex phase diagrams
- Unveil hidden symmetries to explore new states of matter
- Connect seemingly unrelated physical phenomena
- Obtain exact solutions and develop better approximation schemes
- Duality Maps: Strong Coupling to Weak Coupling relations
- Simulation of one system by another physical system
Emergence of a **new paradigm** in Physics of matter, characterized by coexistence and competition of "**non-trivial**" phases.

- **Absence of a small parameter:** Standard mean-field theories do not work
- **Lack of exact solutions**
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Connect Seemingly Unrelated Physical Phenomena

Can one map the Physics of a quantum gas of particles to that of a quantum magnet? Unveiling Order Parameters and Universality

Longitudinal FM  Planar FM  Planar AF  Longitudinal AF

Phase segregation  BEC at $k = 0$  BEC at $k = Q$  CDW at $k = Q$

$Jz/J, V/2t$
Can one map the Physics of a quantum gas of particles to that of a quantum magnet? Unveiling Order Parameters and Universality

Obtain Exact Solutions

\[ H_\theta(J) = J\sqrt{2} \sum_{\langle i,j \rangle} \left[ \cos \theta \mathbf{S}_i \cdot \mathbf{S}_j + \sin \theta (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \right] \]
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What is the connection? — Duality

Majorana wires (Interacting)

XXZ honeycomb (spin 1/2)

Quantum Ising Gauge
What is the connection? — Duality

All share exactly the same spectrum *(unitarily equivalent)*

Majorana wires (Interacting)  XXZ honeycomb (spin 1/2)  Quantum Ising Gauge
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A model requires physical systems that can be controlled by modulating the parameters of the system Hamiltonian.
The Laws of Computation are the Laws of Physics

A model requires physical systems that can be controlled by modulating the parameters of the system Hamiltonian.
The Many Languages and Dictionaries of Nature:
- Bosonic and Hierarchical Languages
- Transmutation of Statistics: Fermionic (Anyonic) Languages
- Building Dictionaries: Ex. Generalized Jordan-Wigner transformations

Some Applications:
- Unveiling Order and Organizing Principles behind Complexity

Quantum Simulations: Computer vs Simulator
- Fermions and Qubits
Problem Setup

State Space:  \[ |\Psi\rangle \in \mathcal{H} \]

\[ \mathcal{H} = \bigotimes_i \mathcal{H}_i \]

\[ \text{dim} \mathcal{H}_i = D \]

Observables:

\[ \hat{O} \ (\hat{O}^\dagger = \hat{O}) \]

Energy, Polarization, \ldots

Dynamics:

\[ H = \sum_{i,j} \langle i|h|j \rangle a_i^\dagger a_j + \frac{1}{2} \sum_{i,j,k,l} \langle ij|g|kl \rangle a_i^\dagger a_j^\dagger a_l a_k \]
Dimensional Reduction - QH Physics

First Quantization

\[ \nu = \frac{N - 1}{L - 1} \]

\[ H_{\text{QH}} = \sum_{i=1}^{N} 2m + \sum_{i<j} V(x_i - x_j) \]

Second Quantization

\[ \hat{P}_{\text{LLL}} H_{\text{QH}} \hat{P}_{\text{LLL}} \]

\[ \hat{H}_{\text{QH}} = \sum_{0<j<L-1} \sum_{k(j),l(j)} V_{j;kl} c_{j+k}^{\dagger} c_{j-k} c_{j-l} c_{j+l} \]

Dynamical momenta

2D continuous geometries

1D orbital lattices
Languages and Dictionaries: Classical Systems

Classical spins to lattice gas or binary alloy mappings

Ising Model: (magnetism)

\( \{ S_i \} \in \text{configuration}; \ S_i = \pm 1 \)

\[
H = J \sum_{\langle i,j \rangle} S_i S_j - B \sum_i S_i
\]

\( N_\uparrow + N_\downarrow = N_s \)

Lattice Gas: (e.g., crystal melting)

Collection of “atoms” (no \( T \))

One atom at each site

\( N_\uparrow \to N_\bullet, \ N_\downarrow \to \text{empty sites} \)

Binary Alloy: (e.g., \( \beta \)-brass)

Two types of “atoms” (no \( T \))

One atom at each site (● or ○)

\( N_\bullet + N_\circ = N_s \)
A **bosonic language** is a set of operators grouped in subsets $S_i$ (associated to each mode $i$) that satisfy the conditions:

- Each $S_i$ is a set of elements $b_i^\mu$ which belong to the algebra of endomorphisms for the vector space $\mathcal{H}_i$ over the field $\mathbb{C}$, $b_i^\mu : \mathcal{H}_i \to \mathcal{H}_i$, and are linearly independent.

- The elements of $S_i$ generate a monoid of linear transformations under the associative product in the algebra which acts irreducibly on $\mathcal{H}_i$ in the sense that the only subspaces stabilized by $S_i$ are $\mathcal{H}_i$ and $\emptyset$.

- If $b_i^\mu$ and $b_j^\nu$ are elements of different subsets $S_i$ and $S_j$, then

$$[b_i^\mu, b_j^\nu] = b_i^\mu b_j^\nu - b_j^\nu b_i^\mu = 0, \quad \text{if } i \neq j$$
\[
1 \leq \alpha, \beta \leq N_f
\]

\[
\begin{align*}
[b_{i\alpha}, b_{j\beta}] &= [b_{i\alpha}^\dagger, b_{j\beta}^\dagger] = 0, \\
[b_{i\alpha}, b_{j\beta}^\dagger] &= \delta_{ij} \delta_{\alpha\beta}, \\
[n_{i\alpha}, b_{j\beta}] &= \delta_{ij} \delta_{\alpha\beta} b_{i\alpha}^\dagger,
\end{align*}
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[\bar{b}_{i\alpha}, \bar{b}_{j\beta}] &= [\bar{b}_{i\alpha}^\dagger, \bar{b}_{j\beta}^\dagger] = 0, \\
[\bar{b}_{i\beta}, \bar{b}_{j\alpha}^\dagger] &= \delta_{ij} (\delta_{\alpha\beta} - \bar{n}_i \delta_{\alpha\beta} - \bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}), \\
[\bar{b}_{i\alpha}^\dagger \bar{b}_{i\beta}, \bar{b}_{j\gamma}^\dagger] &= \delta_{ij} \delta_{\beta\gamma} \bar{b}_{i\alpha}^\dagger,
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[\tilde{b}_{i\alpha}, \tilde{b}_{j\beta}^\dagger] &= \delta_{ij} \delta_{\alpha\beta} (1 - 2 \tilde{n}_\alpha), \\
[\tilde{n}_{i\alpha}, \tilde{b}_{j\beta}] &= \delta_{ij} \delta_{\alpha\beta} \tilde{b}_{i\alpha}^\dagger,
\end{align*}
\]
\begin{align*}
1 \leq \alpha, \beta \leq N_f & \quad N_f = 2 \\
\left\{ \begin{array}{l}
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[\bar{b}^\dagger_{i\alpha} \bar{b}_{i\beta}, \bar{b}^\dagger_{j\gamma}] = \delta_{ij} \delta_{\beta\gamma} \bar{b}^\dagger_{i\alpha} ,
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\left[ \tilde{b}_{i\alpha}, \tilde{b}_{j\beta} \right] &= \left[ \tilde{b}_{i\alpha}^\dagger, \tilde{b}_{j\beta}^\dagger \right] = 0 , \\
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\end{align*}

\text{CCR} \quad D \rightarrow \infty
1 \leq \alpha, \beta \leq N_f

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\end{align*}

\begin{align*}
N_f &= 2 \\
D &= N_f + 1 \\
D \rightarrow \infty
\end{align*}
\[ 1 \leq \alpha, \beta \leq N_f \]

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\end{align*}

CCR

\begin{align*}
D &= N_f + 1, \\
D &= 2^{N_f}
\end{align*}
$1 \leq \alpha, \beta \leq N_f$

\[
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\end{align*}
\]

Physics defines what language is more appropriate
Fundamental Theorem

Given two bosonic languages having the same finite dimension $D$ of their local Hilbert spaces $H_i$, the generators of one of them can be written as a polynomial function of the generators of the other language and vice versa.

(Corollary of Burnside’s Theorem)
Languages and Dictionaries

Example I: Matsubara-Matsuda transformation.

Spin 1/2 Hard Core Boson

\[
\begin{align*}
S_j^+ & = \bar{b}_j^+ \\
S_j^- & = \bar{b}_j \\
S_j^z & = \bar{b}_j^+ \bar{b}_j - \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
[ S_i^{\mu} , S_j^{\nu} ] & = i\delta_{ij} \epsilon_{\mu\nu\lambda} S_i^{\lambda} , \quad \mu, \nu, \lambda = x, y, z
\end{align*}
\]
Example II: Generalization of the Matsubara-Matsuda transformation to $S = 1$

**SU(2) Spin One**

- **S=1/2 hard core bosons**

\[
\begin{align*}
S_j^+ &= \sqrt{2} \left( \bar{b}^\dagger_{j\uparrow} + \bar{b}_{j\downarrow} \right) \\
S_j^- &= \sqrt{2} \left( \bar{b}_{j\uparrow} + \bar{b}^\dagger_{j\downarrow} \right) \\
S_j^z &= \bar{n}_{j\uparrow} - \bar{n}_{j\downarrow}
\end{align*}
\]
Fundamental Theorem

Given two bosonic languages having the same finite dimension $D$ of their local Hilbert spaces $\mathcal{H}_i$, the generators of one of them can can be written as a polynomial function of the generators of the other language and vice versa. (Corollary of Burnside’s Theorem)

Corollary: In each class of bosonic languages there is at least one which is the conjunction of a Lie algebra $S$ and an irreducible rep $\Gamma_S \left( S \wedge \Gamma_S \right)$, i.e., the generators of the bosonic language are generators of the Lie algebra $S_i$ in the rep $\Gamma_S$.

$$S_2 = S$$

$$S_2 = -S$$

$$D = 2S + 1$$

$$\mathcal{L}_i = u(1) \bigoplus su(2)$$

$$S = \bigoplus_i \mathcal{L}_i$$
Hierarchical Language

**Definition:** Any local operator $\hat{O}$ can be written as a linear combination of the generators of the language.

$$\hat{O} = \sum_{i=0}^{NG} \lambda_i G_i$$

- For each class of languages there is always one hierarchical language (HL) whose generators are the identity $I$ and the generators of $\mathfrak{su}(N)$ in the fundamental representation ($D = N$).

- A Hamiltonian operator in the HL becomes **quadratic** in the symmetry generators of the Hierarchical group.

**Example:** $N = 2$ (Two-level system). The Pauli matrices are generators of $\mathfrak{su}(2)$ in the fundamental representation.

$$\hat{O} = \lambda_0 \ I + \lambda_1 \ \sigma_i^x + \lambda_2 \ \sigma_i^y + \lambda_3 \ \sigma_i^z$$
SU(N) Spin-Particle Mappings

\[
S^{\alpha \beta}(j) = \bar{b}_{j\alpha}^\dagger \bar{b}_{j\beta} - \frac{\delta_{\alpha \beta}}{N}
\]

\[
S^{\alpha 0}(j) = \bar{b}_{j\alpha}^\dagger, \quad S^{0\beta}(j) = \bar{b}_{j\beta}
\]

\[
S^{00}(j) = \frac{N_f}{N} - \sum_{\alpha = 1}^{N_f} \bar{n}_{j\alpha} = - \sum_{\alpha = 1}^{N_f} S^{\alpha \alpha}(j),
\]

where \(1 \leq \alpha, \beta \leq N_f\) runs over the set of particle flavors.

\[
[S^{\mu \mu'}(j), S^{\nu \nu'}(j)] = \delta_{\mu \nu} S^{\mu \nu'}(j) - \delta_{\mu \nu'} S^{\nu \mu'}(j).
\]

\[
S(j) = \begin{pmatrix}
\frac{2}{3} - \bar{n}_j & \bar{b}_{j1} & \bar{b}_{j2} \\
\bar{b}_{j1}^{\dagger} & \bar{n}_{j1} - \frac{1}{3} & \bar{b}_{j1}^{\dagger} \bar{b}_{j2} \\
\bar{b}_{j2}^{\dagger} & \bar{b}_{j2}^{\dagger} \bar{b}_{j1} & \bar{n}_{j2} - \frac{1}{3}
\end{pmatrix}
\]
SU(2) Quantum Spins (e.g., $S = 1$)

Quantum Particles (e.g., $s = \frac{1}{2}$)

SU(3) Quantum Spins

"Hierarchical Language"
We will say that a **fermionic language** is the one generated by the creation and anihilation operators for the canonical fermions:

\[
\{ A, B \} = AB + BA
\]

\[
\begin{align*}
\{ c_i^\alpha, c_j^\beta \} &= \{ c_i^\dagger \alpha, c_j^\dagger \beta \} = 0 , \\
\{ c_i^\alpha, c_j^\dagger \} &= \delta_{ij} \delta_{\alpha\beta}
\end{align*}
\]

and any other one which can be obtained by imposing local constraints to the canonical fermions.
Quantum Statistics

Local Pauli Exclusion Principle:

Fractional Exclusion Statistics parameter $p$

Non-local Exchange (Permutation/Braid) Statistics:
Transmutation of Statistics

\[ \theta = 0 \quad \text{“Bosons”} \]
\[ \theta = \pi \quad \text{“Fermions”} \]

**Local Transmutation:**
\[ c_{j\alpha}^\dagger = \tilde{b}_{j\alpha} \hat{T}_{j\alpha} \]
\[ \hat{T}_{j\alpha}^\theta = \exp[i\theta \sum_{\beta < \alpha} \tilde{n}_{j\beta}] \]
\[ \hat{T}_{j\alpha}^{\theta=\pi} = \hat{T}_{j\alpha} \]
\[ \hat{T}_{j\alpha}^2 = 1, \quad \hat{T}_{j\alpha}^\dagger = \hat{T}_{j\alpha} \]

**Non-local Transmutation:**
\[ c_{j\alpha}^\dagger = \tilde{b}_{j\alpha} \hat{T}_{j\alpha}^{\dagger} K_{j} \]
\[ K_{j}^\theta = \exp\left[i\frac{\theta}{\pi} \sum_{l} \omega(l, j) \tilde{n}_l \right] \]
\[ K_{j}^{1d} = \exp[i\pi \sum_{l<j} \tilde{n}_l] \]
Local Transmutation:
Local Transmutation:
Non-local Transmutation:

\[ e^{i\theta} \]

Local Transmutation:

\[ e^{i\theta} \]
Non-local Transmutation:

Local Transmutation:
We have shown how to connect all possible (spin-particle-gauge) languages used in the quantum description of matter, and proved a fundamental theorem that establishes when two languages can be connected through a dictionary.

**Dictionary:**

- **Language A**
  - $D_A$
  - dim of the local Hilbert spaces are equal to $D_B$

- **Language B**

**Relation between modes:**
- Transmutation of Statistics

For each mode (local Hilbert space $H$ of dim $D$):
- A set of operators $S$ acting irreducibly on $H$
- Set of operators $S$
- Statistics between modes
We have shown how to connect all possible (spin-particle-gauge) languages used in the quantum description of matter, and proved a fundamental theorem that establishes when two languages can be connected through a dictionary.

**Defining a Language amounts to defining the State Space**

Relation between modes:

Transmutation of Statistics

For each mode (local Hilbert space $H$ of dim $D$):
- A set of operators $S$ acting irreducibly on $H$

Set of operators $S$ + Statistics between modes

**Dictionary:**

$\begin{align*}
\text{Language } A & \equiv \text{Language } B \\
D_A & = D_B
\end{align*}$

dim of the local Hilbert spaces are equal
Can one connect the different Languages (spin-fermion-boson-gauge)?

Building Dictionaries
Jordan-Wigner Particles

\[
SU(2) \text{ Spin} \quad \text{Hard-core P}
\]

\[
\begin{align*}
\tilde{b}_j^\dagger &= S_j^+ = c_j^\dagger K_j \\
\tilde{b}_j &= S_j^- = K_j^\dagger c_j \\
\tilde{n}_j - \frac{1}{2} &= S_j^z = n_j - \frac{1}{2}
\end{align*}
\]

\[
S_j^+ = \sqrt{2} (\tilde{c}_j^\dagger K_j + K_j^\dagger \tilde{c}_j) \\
S_j^- = \sqrt{2} (K_j^\dagger \tilde{c}_j + \tilde{c}_j^\dagger K_j) \\
S_j^z = \tilde{n}_j^\uparrow - \tilde{n}_j^\downarrow \\
\tilde{c}_j^\dagger = c_j^\dagger (1 - \tilde{n}_j) \quad \sigma = \uparrow, \downarrow
\]
Generalized Jordan-Wigner Transformations

Spin-Flavor Equivalence

Flavors $\mathcal{F}_1$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$S - 1$</th>
<th>$S - 2$</th>
<th>$1$</th>
<th>$0$</th>
<th>$-1$</th>
<th>$-S + 2$</th>
<th>$-S + 1$</th>
<th>$-S$</th>
</tr>
</thead>
</table>

$S_z$

| $S$   | $S - 1$ | $S - 2$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-S + 2$ | $-S + 1$ | $-S$   |

Flavors $\mathcal{F}_{\frac{1}{2}}$

| $S$   | $S - 1$ | $S - 2$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-S + 2$ | $-S + 1$ | $-S$   |

Integer $S$ | Half-odd Integer $S$
Some Applications
Exploiting The Dictionaries
Connecting seemingly unrelated phenomena: Haldane-gap

- Haldane conjecture: Half-odd integer spin chains have a qualitative different excitation spectrum than integer spin chains

\[ H^{S=1}_{\text{xxz}} = \sum_j S_j^z S_{j+1}^z + \Delta \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right) \]

\[ H^{S=1}_{\text{xxz}} = \sum_j (\bar{n}_{j\uparrow} - \bar{n}_{j\downarrow})(\bar{n}_{j+1\uparrow} - \bar{n}_{j+1\downarrow}) + \Delta \sum_{j\sigma} \left( \bar{c}^\dagger_{j\sigma} \bar{c}_{j+1\sigma} + \bar{c}^\dagger_{j\sigma} \bar{c}_{j+1\sigma} + \text{H.c.} \right) \]

- \( H^{S=1}_{\text{xxz}} \) is a \( t-J_z \) model + SC, where \( t = -\Delta \) and \( J_z = 4 \). For \( 8|t| > J_z \), the charge spectrum of the \( t-J_z \) chain is gapless

- In the particle language, the Haldane gap is a superconducting gap
Ferromagnetism and Bose-Einstein Condensation

\( T = 0, \quad J > 0 \)

\[ H_\theta(J) = J \sqrt{2} \sum_{\langle i,j \rangle} \left[ \cos \theta \; \mathbf{S}_i \cdot \mathbf{S}_j + \sin \theta \; (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \right] \]

\( \text{(SPT)} \Downarrow \quad \theta = \frac{5\pi}{4} \)

\[ H_{\frac{5\pi}{4}}(J) = -J \sum_{\langle i,j \rangle, \sigma} \left( \bar{b}_{i\sigma}^\dagger \bar{b}_{j\sigma} + \text{H.c.} \right) - 2J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \]

\[ - 2J \sum_{\langle i,j \rangle} \left( 1 - \frac{\bar{n}_i + \bar{n}_j}{2} + \frac{3}{4} \bar{n}_i \bar{n}_j \right) = -2J \sum_{\langle i,j \rangle} P_s(i,j) \]
Broken Symmetries

$H \frac{5\pi}{4}$ has a global $SU(3)$ symmetry

**BS:** $(N = N_\uparrow + N_\downarrow \leq N_s)$

- $SU(2)$ of spin ($s = \frac{1}{2}$)
- $U(1)$ of charge

Ground State: $|\Psi_0(N, S_z)\rangle = (\tilde{b}_{0\uparrow}^\dagger)^{N_\uparrow}(\tilde{b}_{0\downarrow}^\dagger)^{N_\downarrow}|0\rangle$

Goldstone modes:

\[
\begin{align*}
|\Psi^h_k(N, S_z)\rangle &= \tilde{b}_{k\sigma}|\Psi_0(N, S_z)\rangle & \text{quasihole,} \\
|\Psi^p_k(N, S_z)\rangle &= \tilde{b}_{k\sigma}^\dagger|\Psi_0(N, S_z)\rangle & \text{quasiparticle}
\end{align*}
\]
Hierarchical Language

\[ H_\theta(J) = J \sqrt{2} \sum_{\langle i,j \rangle} \left[ \cos \theta \mathbf{S}_i \cdot \mathbf{S}_j + \sin \theta (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \right] \]

\[ \downarrow (SU(3) \text{ SPT}) \]

\[ H_\theta(J) = J \sqrt{2} \sum_{\langle i,j \rangle} \left[ \cos \theta S^{\mu\nu}(i)S^{\nu\mu}(j) + (\sin \theta - \cos \theta) S^{\mu\nu}(i)\tilde{S}^{\nu\mu}(j) \right] \]
Fundamental and Conjugate representations

\[
\mathcal{S}(j) = \begin{pmatrix}
\frac{2}{3} - \bar{n}_j & \bar{b}_{j\uparrow} & \bar{b}_{j\downarrow} \\
\bar{b}_{j\uparrow} & \bar{n}_{j\uparrow} - \frac{1}{3} & \bar{b}_{j\uparrow} \bar{b}_{j\downarrow} \\
\bar{b}_{j\downarrow} & \bar{b}_{j\downarrow} \bar{b}_{j\uparrow} & \bar{n}_{j\downarrow} - \frac{1}{3}
\end{pmatrix}
\]

“Quark”

\[
\tilde{\mathcal{S}}(j) = \begin{pmatrix}
\frac{2}{3} - \bar{n}_j & -\bar{b}_{j\downarrow} & -\bar{b}_{j\uparrow} \\
-\bar{b}_{j\downarrow} & \bar{n}_{j\downarrow} - \frac{1}{3} & \bar{b}_{j\uparrow} \bar{b}_{j\downarrow} \\
-\bar{b}_{j\uparrow} & \bar{b}_{j\downarrow} \bar{b}_{j\uparrow} & \bar{n}_{j\uparrow} - \frac{1}{3}
\end{pmatrix}
\]

“Anti – Quark”

\[
[S^{\mu\mu'}(j), S^{\nu\nu'}(j)] = \delta_{\mu'\nu}S^{\mu\nu'}(j) - \delta_{\mu\nu'}S^{\nu\mu'}(j)
\]
## Global $SU(3)$ Order Parameters

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Global $SU(3)$ OP</th>
<th>OP 1</th>
<th>OP 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5\pi/4$ (FM-UN)</td>
<td>$S = \sum_j S(j)$</td>
<td>$M = \sum_j S_j$</td>
<td>$N = \sum_j N_j$</td>
</tr>
<tr>
<td>$\pi/4$ (AF-SN)</td>
<td>$S_{ST} = \sum_j S(j)e^{iQ \cdot r_j}$</td>
<td>$M_{ST} = \sum_j S_j e^{iQ \cdot r_j}$</td>
<td>$N_{ST} = \sum_j N_j e^{iQ \cdot r_j}$</td>
</tr>
<tr>
<td>$3\pi/2$ (AF-UN)</td>
<td>$S_+ = \sum_{j \in A} S(j) + \sum_{j \in B} \hat{S}(j)$</td>
<td>$M_{ST} = \sum_j S_j e^{iQ \cdot r_j}$</td>
<td>$N = \sum_j N_j$</td>
</tr>
<tr>
<td>$\pi/2$ (FM-SN)</td>
<td>$S_- = \sum_{j \in A} S(j) - \sum_{j \in B} \hat{S}(j)$</td>
<td>$M = \sum_j S_j$</td>
<td>$N_{ST} = \sum_j N_j e^{iQ \cdot r_j}$</td>
</tr>
</tbody>
</table>
Finding useful translations: The one-dimensional Hubbard model

\[
H_{\text{Hubb}}^{1d} = t \sum_{j, \sigma} (c_{j \sigma}^\dagger c_{j+1 \sigma} + c_{j+1 \sigma}^\dagger c_{j \sigma}) + U \sum_{j=1}^{N} (\hat{n}_{j \uparrow} - \frac{1}{2}) (\hat{n}_{j \downarrow} - \frac{1}{2})
\]

Bond \(j\) ↔ Site \(j\)

\[
\left\{
\begin{array}{l}
S_{j1}^+ = c_{j1}^\dagger \bar{K}_{j\uparrow}, \\
S_{j1}^z = \hat{n}_{j\uparrow} - \frac{1}{2}, \\
S_{j2}^+ = c_{j2}^\dagger \bar{K}_{j\downarrow}, \\
S_{j2}^z = \hat{n}_{j\downarrow} - \frac{1}{2},
\end{array}
\right.
\]

\(\bar{K}_{j\uparrow} = \exp[i\pi (\sum_{1} \hat{n}_{1\downarrow} + \sum_{1<j} \hat{n}_{1\uparrow})]\)

\(\bar{K}_{j\downarrow} = \exp[i\pi \sum_{1<j} \hat{n}_{1\downarrow}]\)

\[
H_{\text{Hubb}}^{1d} = 2t \sum_{j, \nu} (S_{j\nu}^x S_{j+1\nu}^x + S_{j\nu}^y S_{j+1\nu}^y) + U \sum_{j=1}^{N} S_{j1}^z S_{j2}^z
\]
Finding useful translations: The one-dimensional Hubbard model

\[
H_{\text{Hubb}}^{1d} = t \sum_{j, \sigma} (c_{j+1}^{\dagger} c_{j+1}^{\sigma} + c_{j+1}^{\sigma} c_{j}^{\dagger}) + U \sum_{j=1}^{N} (\hat{n}_{j}^{\uparrow} - \frac{1}{2}) (\hat{n}_{j}^{\downarrow} - \frac{1}{2})
\]

\[
S_{j1}^{+} = c_{j1}^{\dagger} \tilde{K}_{j}^{\uparrow}, \quad S_{j1}^{-} = \hat{n}_{j}^{\uparrow} - \frac{1}{2}, \quad \tilde{K}_{j}^{\uparrow} = \exp[i \pi (\sum_{1} \hat{n}_{1}^{\downarrow} + \sum_{1<j} \hat{n}_{1}^{\uparrow})]
\]

\[
S_{j2}^{+} = c_{j1}^{\dagger} \tilde{K}_{j}^{\downarrow}, \quad S_{j2}^{-} = \hat{n}_{j}^{\downarrow} - \frac{1}{2}, \quad \tilde{K}_{j}^{\downarrow} = \exp[i \pi \sum_{1<j} \hat{n}_{1}^{\downarrow}]
\]

\[
H_{\text{Hubb}}^{1d} = \sum_{j} [J_{\mu \nu} S_{\mu \nu}^{\mu} (j) S_{\nu \mu}^{\nu} (j + 1) + J'_{\mu \nu} S_{\mu \nu}^{\mu} (j) \tilde{S}_{\nu \mu}^{\nu} (j + 1) + \frac{U}{2} (S_{00}^{00} (j) + S_{33}^{33} (j))]
\]

\[
S (j) = \begin{pmatrix}
S_{00}^{00} (j) & \tilde{b}_{j}^{\uparrow} (1 - \tilde{n}_{j}^{\downarrow}) & b_{j}^{\uparrow} (1 - \tilde{n}_{j}^{\downarrow}) & \tilde{b}_{j}^{\downarrow} \tilde{b}_{j}^{\downarrow} \\
(1 - \tilde{n}_{j}^{\downarrow}) b_{j}^{\downarrow} & S_{11}^{11} (j) & \tilde{b}_{j}^{\uparrow} b_{j}^{\downarrow} & \tilde{n}_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} \\
(1 - \tilde{n}_{j}^{\downarrow}) \tilde{b}_{j}^{\uparrow} b_{j}^{\downarrow} & \tilde{b}_{j}^{\uparrow} \tilde{b}_{j}^{\downarrow} & S_{22}^{22} (j) & \tilde{n}_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} \\
\tilde{b}_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} & \tilde{b}_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} & \tilde{b}_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} & S_{33}^{33} (j)
\end{pmatrix}, \quad \tilde{S} (j) = \begin{pmatrix}
-\tilde{S}_{00}^{00} (j) & -\tilde{b}_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} & -b_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} & -\tilde{b}_{j}^{\downarrow} \tilde{b}_{j}^{\downarrow} \\
-\tilde{n}_{j}^{\uparrow} \tilde{b}_{j}^{\uparrow} & -S_{11}^{11} (j) & -b_{j}^{\uparrow} b_{j}^{\downarrow} & -\tilde{n}_{j}^{\uparrow} b_{j}^{\downarrow} \\
b_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} & -b_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} & -\tilde{b}_{j}^{\downarrow} b_{j}^{\downarrow} & -S_{22}^{22} (j) \\
-\tilde{b}_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} & -\tilde{b}_{j}^{\downarrow} \tilde{n}_{j}^{\downarrow} & \tilde{b}_{j}^{\uparrow} \tilde{n}_{j}^{\downarrow} - 1 & -\tilde{S}_{33}^{33} (j)
\end{pmatrix}
\]
## Equivalence between Spin-Particle Models

### Concept of Universality (or Equivalence)

<table>
<thead>
<tr>
<th>$d$</th>
<th>Model A</th>
<th>Model B</th>
<th>M - S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Isotropic $XY$ ($S = 1/2$)</td>
<td>$t$</td>
<td>C - E</td>
</tr>
<tr>
<td>1</td>
<td>Anisotropic $XY$ ($S = 1/2$)</td>
<td>$t - \Delta$</td>
<td>C - E</td>
</tr>
<tr>
<td>1</td>
<td>$XXZ$ ($S = 1/2$)</td>
<td>$t - V$</td>
<td>C - E (BA)</td>
</tr>
<tr>
<td>1</td>
<td>$XXZ$ ($S = 1/2$)</td>
<td>$t - J_z$</td>
<td>P - E (BA)</td>
</tr>
<tr>
<td>1</td>
<td>$XYZ$ ($S = 1/2$)</td>
<td>$t - \Delta - V$</td>
<td>C - E (BA)</td>
</tr>
<tr>
<td>1</td>
<td>Majumdar-Ghosh ($S = 1/2$)</td>
<td>$t - t' - V - V'$</td>
<td>C - GS</td>
</tr>
<tr>
<td>1</td>
<td>BB $S = 1$ ($\phi = \pi/4$, LS)</td>
<td>$t - J$</td>
<td>C - E (BA)</td>
</tr>
<tr>
<td>1</td>
<td>BB $S = 1$ ($\tan \phi = 1/3$, AKLT)</td>
<td>$t - \Delta - J - V - \mu$</td>
<td>C - GS</td>
</tr>
<tr>
<td>1</td>
<td>BB $S = 1$ ($\phi = 7\pi/4$, TB)</td>
<td>$t - \Delta - J - V - \mu$</td>
<td>C - E (BA)</td>
</tr>
<tr>
<td>1</td>
<td>BB $S = 1$ ($\phi = 3\pi/2$, K)</td>
<td>$\Delta - J - V - \mu$</td>
<td>C - QE</td>
</tr>
<tr>
<td>1</td>
<td>$S = 3/2$</td>
<td>F ($S = 1/2$) Hubbard</td>
<td>C - E (BA)</td>
</tr>
<tr>
<td>2</td>
<td>$U(1)$ gauge magnet</td>
<td>F strings</td>
<td>C - E</td>
</tr>
<tr>
<td>2</td>
<td>Shastry-Sutherland ($S = 1/2$)</td>
<td>$t - t' - V - V'$</td>
<td>C - GS</td>
</tr>
<tr>
<td>Any</td>
<td>BB $S = 1$ ($\phi = 5\pi/4$)</td>
<td>$t - J - V - \mu$</td>
<td>C - QE</td>
</tr>
<tr>
<td>Any</td>
<td>Anisotropic $S = 1$ Heisenberg</td>
<td>B ($S = 1/2$) Hubbard</td>
<td>P - U</td>
</tr>
</tbody>
</table>
Classification of local Order Parameters

The generators of the HL exhaust all possible OPs

**Recipe:**
Identify the group of symmetries of H, G
Classify the generators of the HL according to the irreps of G
Each irrep leads to a different broken symmetry OP

Hierarchical mean-field theories

All OPs are treated on an equal footing
Help to design/engineer new states of matter

Concept of Emergent Symmetry

Helps to identify the relevant low energy degrees of freedom (bricks to build new phases)
Hierarchical Mean-field of the Heisenberg model in Kagome

\[ t/V = -0.5 \]

Ising

\[ \mu = 2/3 \]

QMC
DMRG
iPEPS
36-ED
9-HMFT
27-HMFT

VBC
VBC_{1/9}
VBC_{2/3}
VBC_{5/9}
VBC_{7/9}
VBC_{8/9}
VBC_{14/27}
VBC_{16/27}
VBC_{17/27}
VBC_{19/27}
VBC_{20/27}
VBC_{23/27}

\[ \pi VBC \]
\[ \pi VBC_{2/3} \]
\[ \pi VBC_{5/9} \]

CSF

\[ \rho = 2/3 \]

SF

\[ t' \]
Simulation of Physical Phenomena

Dictionaries: Fermions and Bosons to Qubits
Equivalence between Models of Computation

Fermion Computation via the Standard (Qubit) Model

If it is possible to efficiently simulate the fermion model by the standard model then these two models of computation are equivalent.

Prove that every step can be done with the same complexity.

The two models are equivalent.

(Tool: Generalized Jordan-Wigner mappings)
Equivalence between Models of Computation

Fermion Computation via the Standard (Qubit) Model

If it is possible to efficiently simulate the fermion model by the standard model, then these two models of computation are equivalent.

\[
\begin{array}{c}
\text{Preparation} \\
\text{Initial State} \\
\text{Evolution} \\
\text{Statistics} \\
\Psi_0 \\
\Psi_t \\
\end{array}
\]

Prove that every step can be done with the same complexity.

Noise control

Unitary

Dynamical Correlation

Exponential Speed-up

\( e^{\frac{n}{2}} \)

The two models are equivalent.

Superpolynomial Speed-up

\( (e^n) \)

Grover’s Search

Shor’s Factoring

Functions

Dynamical Correlation

On
Example: Resonant Impurity Scattering

Physical System: \[ L = n a \ , \ R_i = i a \ , \ (c^\dagger_{n+1} = c^\dagger_1) \]

\[ H = -\mathcal{T} \sum_{i=1}^{n} (c^\dagger_i c_{i+1} + c^\dagger_{i+1} c_i) + \epsilon b^\dagger b + \frac{V}{\sqrt{n}} \sum_{i=1}^{n} (c^\dagger_i b + b^\dagger c_i) \]

Phenomenon to study: Probability to stay in \( |\Psi(0)\rangle \)

\[ G(t) = \langle \Psi(0) | b(t) b^\dagger(0) |\Psi(0)\rangle \ , \ b(t) = e^{iHt} b(0) e^{-iHt} \]

Initial state: Fermi sea of \( N_e \leq n \) fermions

\[ |\Psi(0)\rangle = |\text{FS}\rangle = \prod_{i=0}^{N_e-1} c^\dagger_{k_i} |0\rangle \ , \ c^\dagger_{k_i} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{ik_i R_j} c^\dagger_j \]

\[ \mathcal{E}_{k_i} = -2\mathcal{T} \cos k_i a \]
Quantum Algorithm to Compute $G(t)$

### Spin-Fermion mapping:

$b = \sigma^1_-$

$c_{k0} = -\sigma_z^1 \sigma^-_z$

$\vdots$

$c_{kn-1} = (-1)^n \sigma_z^1 \sigma_z^2 \cdots \sigma_z^n \sigma_-^{n+1}$

$b^\dagger = \sigma^1_+$

$c_{k0}^\dagger = -\sigma_z^1 \sigma^+_z$

$\vdots$

$c_{kn-1}^\dagger = (-1)^n \sigma_z^1 \sigma_z^2 \cdots \sigma_z^n \sigma_+^{n+1}$

### Physical property: Effective 2-Qubit problem

$G(t) = \langle W_b \rangle = \langle e^{i\bar{H}t} \sigma^1_+ e^{-i\bar{H}t} \sigma^1_- \rangle$

$\bar{H} = \frac{\epsilon}{2} \sigma^1_z + \frac{\mathcal{E}_{k0}}{2} \sigma^2_z + \frac{V}{2} (\sigma^1_x \sigma^2_x + \sigma^1_y \sigma^2_y)$

### Translation to NMR-machine language:

$e^{-i\bar{H}t} = U e^{-iH_{P1}t} U^\dagger$, $H_{P1} = h_- \sigma^1_z + h_+ \sigma^2_z$

$U = e^{i\frac{\pi}{4} \sigma^2_x} e^{-i\frac{\pi}{4} \sigma^1_y} e^{-i\frac{\theta}{2} \sigma^1_z \sigma^2_z} e^{i\frac{\pi}{4} \sigma^1_y} e^{i\frac{\pi}{4} \sigma^1_x} e^{-i\frac{\pi}{4} \sigma^2_x} e^{-i\frac{\pi}{4} \sigma^1_y} e^{i\frac{\theta}{2} \sigma^1_z \sigma^2_z} e^{-i\frac{\pi}{4} \sigma^1_x} e^{i\frac{\pi}{4} \sigma^2_y}$

**NMR-pulses**
\[ \epsilon = -8, \; \mathcal{E}_0 = -2, \; V = 4 \]

\[ \epsilon = 0, \; \mathcal{E}_0 = -2, \; V = 4 \]

\[ \epsilon = -8, \; \mathcal{E}_0 = -2, \; V = 1/2 \]

Green’s Function: \( G(t) \)

C. Negrevergne, R. Somma, G.O., M. Knill, R. Laflamme

Energy Spectrum: Fourier-transform
Simulating Number-conserving Canonical Bosons with Spins

\[ N_P = 3 \]

\[
\hat{b}^\dagger = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \sqrt{2} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \sqrt{N_P} & 0 \\
\end{pmatrix}
\]

\[ (\hat{b}^\dagger)^{N_P+1} = 0 \rightarrow \text{up to } N_P \text{ bosons} \]

\[ \bar{b}^\dagger_i = 1 \otimes \cdots \otimes 1 \otimes \hat{b}^\dagger \otimes 1 \otimes \cdots \otimes 1 \]

\[ i^{th} \text{ factor} \]

\[ |\phi_\alpha\rangle = \frac{1}{2\sqrt{3}} b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger |\text{vac}\rangle \]

\[ |\phi_\alpha\rangle = \left| \uparrow \downarrow \uparrow \downarrow \uparrow \right>_1 \otimes \left| \downarrow \uparrow \downarrow \uparrow \uparrow \right>_2 \otimes \left| \uparrow \uparrow \downarrow \uparrow \downarrow \right>_3 \otimes \left| \downarrow \downarrow \downarrow \downarrow \uparrow \right>_4 \otimes \left| \uparrow \uparrow \uparrow \uparrow \uparrow \right>_5 \]

\[ \bar{b}^\dagger_i \left| n \right>_i = \sqrt{n + 1} \left| n + 1 \right>_i \]

\[ \bar{b}^\dagger_i = \sum_{n=0}^{N_P-1} \sqrt{n + 1} \sigma^{-} \sigma^{n+1, i} \]