2 Fermi Liquids

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1 A tale of many fermions

One of the triumphs of quantum mechanics in the 20th century was the explanation of the atomic structure (periodic table of the elements) in terms of elementary fermions (electrons) endowed with intrinsic spin $\hbar/2$ and obeying Pauli's exclusion principle, such that no two fermions can occupy the same quantum state. When many fermions (N) are present in a finite region of space the Pauli exclusion principle creates a kind of effective repulsion between the particles: this repulsion exists even if we pretend that there are no physical interactions between the particles, e.g., in the limiting case of the ideal Fermi gas. Under such conditions the minimum energy state is attained by singly occupying the N lowest lying single-particle states: this maximally compact state is known (for large N) as the "Fermi sea", and the energy of the highest occupied state is known as the Fermi energy, denoted by ϵ_F. When many interacting fermions condense in a state close to the minimum energy allowed by the Pauli exclusion principle (i.e., such that the average occupation numbers of the N lowest single-particle states are close to 1) one obtains a *Fermi liquid*. Physical realizations of the Fermi liquids concept range from interacting electrons in metals and semiconductors (our main interest here), to liquid ^{3}He, to gases of cold fermionic atoms, to nuclear matter, electrons in white dwarves, and neutron stars.

The behavior of Fermi liquids confronts us at the outset with a puzzle. In spite of strong mutual interactions the particles appear in many measurements to behave as if they were essentially non-interacting. Many properties of electrons in metals, for example heat capacity and electric conductivity, can be qualitatively understood in terms of the Sommerfeld picture, which is based on the degenerate ideal Fermi gas model. But this model is not easily justified. The average interaction energy per electron can be roughly estimated to be on the order of $e^2 n^{\frac{1}{3}}$, where $n = N/V$ is the average electronic density and V is the volume. For electrons in Na this works out to be $\simeq 4.29$ eV, which is already larger than the value $\epsilon_F \simeq 3.24$ eV of the Fermi energy. There is no sense in which the Coulomb interaction can be considered a small perturbation, and yet the electrons behave by and large as noninteracting particles, while the presence of the interaction manifests itself in rather subtle ways.

It was not until the late 1950s that this puzzling state of affairs was clarified from a theoretical point of view by L.D. Landau [1]. Although Landau did not provide a rigorous solution of the problem, he did provide a firm basis for the understanding of the "normal" low-energy behavior of interacting Fermi systems. The solidity of this basis was subsequently confirmed by theoretical and experimental work.

Perhaps the most striking feature of Landau's approach is that he completely sidestepped what most physicists would have considered a prerequisite for further progress, namely a complete description of the interacting ground state. Instead, he focused on the excited states. His basic idea was that, under very broad conditions, the low-lying excitations of a system of interacting Fermions with repulsive interactions are connected to the low-lying states of a non-interacting ideal Fermi liquid by a suitably slow switching-on of the interaction between the particles. There are several subtleties in the specification of the switching-on process, beginning with a precise definition of the words "suitably slow". For the time being we will not delve into these

subtleties, but simply notice that the switching-on procedure establishes a one-to-one correspondence between the eigenstates of the ideal system and a set of (approximate) eigenstates of the interacting system. Since the eigenstates of the noninteracting system are specified by a set of occupation numbers $\{\mathcal{N}_{\vec{k}\sigma}\}$ of single-particle momentum eigenstates, it follows that the corresponding low-lying excitations of the interacting system can also be described by the same set of occupation numbers. It is important to understand that the quantum numbers $\mathcal{N}_{\vec{k}\sigma}$ that specify an excited state of the Fermi liquid are not the true momentum occupation numbers $n_{k\sigma} = \langle \hat{a}^\dagger_{\vec{k}\sigma} \hat{a}_{\vec{k}\sigma} \rangle$ for that state. Rather, they are momentum occupation numbers of the ideal system from which the excited state has evolved.

Because in an interacting system the $n_{\vec{k}\sigma}$'s are not constants of the motion, one could have naively expected that any memory of the initial noninteracting momentum distribution $\{\mathcal{N}_{\vec{k}\sigma}\}$ would be completely lost at the end of the switching-on process. Landau's insight was the recognition that, for states that are weakly excited (i.e., close to the noninteracting Fermi distribution) the occupation numbers change very slowly even when particle-particle interactions are strong. The main consequence of this fact is that the $\mathcal{N}_{\vec{k}\sigma}$ retain their validity as approximate quantum numbers, which specify an excited state. Thus, low-energy elementary excitations of an interacting Fermi liquid can be described in terms of addition or removal of individual quasiparticles from a filled Fermi sphere of radius k_F, where k_F is the Fermi momentum of a non interacting electron gas of the same density. In other words, the interacting system has a Fermi surface that coincides with that of the non-interacting system – a statement that is known as *Luttinger's theorem*. For example, a state of the ideal system containing a particle of momentum $\hbar\vec{k}$ with $k \geq k_F$ outside the non-interacting Fermi sphere evolves into an excited state of the interacting system containing one quasiparticle of momentum $\hbar\vec{k}$ outside the interacting Fermi sphere. Likewise, a state of the noninteracting system containing one empty state (a hole) of momentum $\hbar\vec{k}$ within the non-interacting Fermi sphere evolves into an excited state of the interacting system containing a quasihole of the same momentum. More complex excitations consisting of multiple quasiparticles and quasiholes can be constructed in a similar fashion. By definition, the ground-state has $\mathcal{N}_{\vec{k}\sigma} = \Theta(k_F - k)$. In contrast to this, the momentum occupation numbers $n_{\vec{k}\sigma}$ in the ground-state decrease discontinuously by an amount Z (with $0 < Z \leq 1$) as $\vec{k}$ crosses the Fermi sphere from inside to outside. The discontinuity, Z, is of course 1 in the noninteracting system and less than 1 in the interacting system. The existence of a discontinuity in the momentum occupation number at $k = k_F$ is one of the distinctive signatures of the normal Fermi liquid.

The physical basis of the Landau theory rests on the surprising ineffectiveness of electron-electron scattering to change the momentum distribution of quasiparticles near the Fermi level. What happens is that most of the states into which two quasiparticles near the Fermi surface might end up after a collision are already occupied by other electrons, and therefore, according to the Pauli exclusion principle, unavailable. Because of this "Pauli blocking" effect, which operates irrespective of the strength of the interaction, the rate at which a quasiparticle is scattered out of a state of momentum $k \simeq k_F$ vanishes for $k \to k_F$. This result can be obtained from a simple phase space argument.

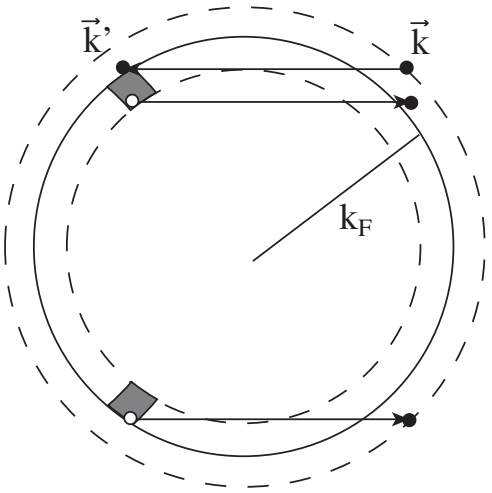

Fig. 1: *Schematic illustration of two possible decay processes for a quasiparticle near the Fermi surface. The quasiparticle makes a transition from $\vec{k}$ to $\vec{k}'$, producing an electron-hole pair in the process. Momentum and energy conservation restrict the momentum of the hole to the shaded regions.*

Consider a quasiparticle with initial wave vector $\vec{k}$ with $k > k_F$. At zero temperature the empty states into which the quasiparticle can decay lie within a shell of thickness $|k-k_F|$ just above the Fermi surface. The number of states in this region is clearly proportional to $|k-k_F|$ – a result valid in one, two and three dimensions. Now, through the Coulomb interaction, the momentum and energy change of the quasiparticle will be offset by the momentum and energy of an electron-hole pair. In two and three dimensions, the state of the hole must lie within a shell of thickness $|k - k_F|$ below the Fermi surface (see Fig. 1). This contributes another factor of $|k-k_F|$ to the probability of decay, which, as anticipated, is thus found to be proportional to $(k-k_F)^2$ in three dimensions.[1]

Accordingly in two and three dimensions in the limit $k \to k_F$, the inverse of the scattering rate, i.e., the scattering time $\tau_{\vec{k}}$, becomes long enough to include many cycles of oscillation of an external field that is able to create the quasiparticle excitation out of the ground-state (the frequency of this field being proportional to the excitation energy which is of order $|k-k_F|$). Thus, on a time scale that is short compared to τ_k (but still long compared to the inverse excitation frequency) the occupation number $\mathcal{N}_{\vec{k}\sigma}$ can be regarded as a good quantum number for the excited state.[2]

Strictly speaking however the $\mathcal{N}_{\vec{k}\sigma}$ are not exact quantum numbers (for that to be true the scattering rate would have to actually vanish), and if one waits long enough, i.e., up to times $t \gg \tau_{\vec{k}}$,

[1] In two dimensions, a more accurate calculation shows that the scattering rate vanishes at a somewhat slower rate $-(k-k_F)^2 \ln|k-k_F|$ (see Section 3.3). In the presence of disorder the quasiparticle inelastic lifetime is shorter. This is discussed in Section 6.1.

[2] The alert reader will notice that the phase-space argument is circular: one assumes the existence of quasiparticles to deduce that their lifetime is long. This does not prove the existence of quasiparticles, but shows that one can assume their existence without falling in a contradiction. This discussion does also clarify that a "suitably slow" switching-on must be carried out in a time intermediate between the fast time scale of order $\frac{1}{v_F|k-k_F|}$ related to the resolution of a quasiparticle state and the quasiparticles lifetime which we have shown to be of order $\frac{\epsilon_F}{\hbar(v_F|k-k_F|)^2}$.

one will see them change. Thus a quasiparticle state is *not* an exact eigenstate of the interacting Fermi liquid. Rather, it must be understood as a superposition of closely spaced exact eigenstates with energies spread over a range of width $\hbar/\tau_{\vec{k}} \ll \varepsilon_{\vec{k}}$ about the median quasi-particle energy $\varepsilon_{\vec{k}}$. Such a state decays with a characteristic lifetime $\tau_{\vec{k}}$, and can be regarded as stationary only for times much shorter than $\tau_{\vec{k}}$. It turns out that these "quasi-eigenstates" completely determine the response of the system to macroscopic perturbations.

So strong has been the impact of the Landau theory in condensed matter physics that the Landau Fermi liquid is often referred to as the Fermi liquid *tout-court*. It is important therefore to keep in mind that there are situations in which the Landau description of the Fermi liquid fails completely. A well known case is that of one-dimensional electronic systems. In this case the spectral density of electron-hole pairs exhibits a peak at small wavevector q and frequency $\omega = v_F q$. All the electron-hole pairs with given momentum $\hbar q$ have essentially the same energy comprised between $\hbar|\omega_-(q)| = \left|\hbar v_F q - \frac{\hbar^2 q^2}{2m}\right|$ and $\hbar\omega_+(q) = \hbar v_F q + \frac{\hbar^2 q^2}{2m}$. Due to the massive quasi-degeneracy of the noninteracting spectrum, an arbitrarily weak interaction causes a complete reconstruction of the many-body eigenstates, and, in particular, destroys the "Fermi surface" (actually, two points in 1D). Thus, there is no Landau Fermi liquid in one dimension. The correct paradigm in this case is the *Luttinger liquid*, where the low-energy excitations are collective charge and spin density waves and single-particle excitations are no longer sharply defined. Whether similar departures from Fermi liquid theory can also occur in two dimensions remains a very active and contentious area of research.

In spite of its occasional failures, the concept of a quasiparticle is an encompassing one. It applies not only to electrons in metals and doped semiconductors (where the renormalization factors $\frac{m^*}{m}$ and Z remain close to 1) but also to ^{3}He atoms in the liquid phase where $\frac{m^*}{m} \simeq 3$ and to highly correlated heavy fermion systems where $\frac{m^*}{m}$ can run in the hundreds. Considering the diversity in coupling constants and physical character of these systems it is quite amazing that they can be subsumed under the same generic theoretical paradigm.

This Chapter is organized as follows:

In Section 2 we summarize the basic results of the phenomenological theory of the Fermi liquids due to and mostly developed by Landau.

Section 3 presents a simple theory of the quasiparticle inelastic lifetime based on the Fermi golden rule of elementary quantum mechanics.

The microscopic underpinning of the Landau theory is briefly reviewed in Section 4 and some numerical results for quasiparticle properties are presented.

Section 5 presents the Fermi liquid theory for massless Dirac Fermions in graphene.

Finally, Section 6 illustrates some ways in which the standard theory of Fermi liquids may fail due to reduced dimensionality, disorder effects, and high magnetic fields.

The material presented in this chapter is largely adapted from the discussion of the normal Fermi liquid theory in "Quantum theory of the electron liquid" (G.F. Giuliani and G. Vignale, Cambridge University Press, 2005) [2] and in "The Theory of Quantum Liquids, Vol. I" (D. Pines and P. Noziéres, W.A. Benjamin 1966) [3], to which you are referred for more detailed derivations and discussions.

2 Phenomenological theory

2.1 The Landau energy functional

As discussed in the previous section, a set of noninteracting occupation numbers $\mathcal{N}_{k\sigma} = 0$ or 1 defines, by continuation, a quasi-eigenstate of the interacting Fermi liquid. Similarly, a distribution of fractional occupation numbers $0 \leq \mathcal{N}_{k\sigma} \leq 1$ defines an ensemble of quasiparticle states in which the state of momentum $\hbar\vec{k}$ and spin σ has a probability $\mathcal{N}_{k\sigma}$ to be occupied and $1-\mathcal{N}_{k\sigma}$ to be empty. At the heart of Landau's macroscopic theory of the Fermi liquids lies an *Ansatz* for the functional dependence of the energy of the liquid on the quasiparticle distribution function $\mathcal{N}_{\vec{k}\sigma}$. This functional is in fact an expansion for the energy to second order in the deviation of the quasiparticle distribution function from its ground-state value $\mathcal{N}^{(0)}_{\vec{k}\sigma} = \Theta(k_F - k)$:

$$E\big[\{\mathcal{N}_{\vec{k}\sigma}\}\big] = E_0 + \sum_{\vec{k}\sigma} \mathcal{E}_{\vec{k}\sigma}\, \delta\mathcal{N}_{\vec{k}\sigma} + \frac{1}{2}\sum_{\vec{k}\sigma,\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\, \delta\mathcal{N}_{\vec{k}\sigma}\delta\mathcal{N}_{\vec{k}'\sigma'}\ , \tag{1}$$

where E_0 is the ground-state energy (which needs not be specified!), $\mathcal{E}_{\vec{k}\sigma}$ is the energy of a single quasiparticle, $f_{\vec{k}\sigma,\vec{k}'\sigma'}$ is the Landau interaction function and $\delta\mathcal{N}_{\vec{k}\sigma} = \mathcal{N}_{\vec{k}\sigma} - \mathcal{N}^{(0)}_{\vec{k}\sigma}$ is the deviation of the quasiparticle distribution function from the ideal Fermi distribution at $T = 0$.
Because the quasiparticles are well defined only in the immediate vicinity of the Fermi surface, it is evident that this expansion makes sense only when $\delta\mathcal{N}_{\vec{k}\sigma}$ is restricted to a thin shell of momentum space surrounding the Fermi surface. In addition, since every wave vector sum introduces a factor L^d (the d-dimensional volume), the interaction function $f_{\vec{k}\sigma,\vec{k}'\sigma'}$ must scale as the inverse of the volume $\frac{1}{L^d}$ in order to keep the energy proportional to the volume in the thermodynamic limit.
Both the quasiparticle energy and the interaction function (as well as the Landau parameters introduced below) are phenomenological quantities that the Landau theory of Fermi liquid assumes to be given. In practice, they must be either determined from measurements of physical properties, or calculated by a microscopic many-body theory.
The energy $\mathcal{E}_{\vec{k}\sigma}$ of a single quasiparticle can be formally viewed as the functional derivative of the energy with respect to the quasiparticle distribution function evaluated at the ground-state:

$$\mathcal{E}_{\vec{k}\sigma} = \left(\frac{\delta E}{\delta\mathcal{N}_{\vec{k}\sigma}}\right)_{\mathcal{N}_{\vec{k}\sigma}=\mathcal{N}^{(0)}_{\vec{k}\sigma}}\ . \tag{2}$$

Since the ground-state of the $N+1$-particle system is obtained by adding a quasiparticle of wavevector k_F to the ground state of the N-particle system, it is evident that

$$\mathcal{E}_{k_F\sigma} = \mu\ , \tag{3}$$

where μ is the chemical potential.
In an isotropic liquid, for $|\vec{k}|$ close to k_F, the quasiparticle energy can be expanded as

$$\mathcal{E}_{\vec{k}\sigma} \simeq \mu + \hbar v^*_F\,(k - k_F), \tag{4}$$

where

$$v_F^\star \;=\; \frac{1}{\hbar}\left|\frac{\partial \mathcal{E}_{\vec{k}\sigma}}{\partial \vec{k}}\right|_{k=k_F} \tag{5}$$

defines the effective Fermi velocity of a quasiparticle. v_F^* can be conveniently written as

$$v_F^* \;=\; \frac{\hbar k_F}{m^\star}\;, \tag{6}$$

which defines the quasiparticle effective mass $m^\star$. The effective mass in turn determines $N^\star(0)$, the quasiparticle density of states (per unit volume) at the Fermi level μ. This is given by $N^\star(0) \;=\; \frac{m^\star}{m}N(0)$, where $N(0)$ is the density of states at the Fermi surface of a non interacting electron gas, $N(0) = \frac{nd}{2\epsilon_F}$ in d-dimensions.

A fundamental role in the Landau theory is played by the quantity

$$\tilde{\mathcal{E}}_{\vec{k}\sigma} = \frac{\delta E}{\delta \mathcal{N}_{\vec{k}\sigma}} = \mathcal{E}_{\vec{k}\sigma} + \sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta\mathcal{N}_{\vec{k}'\sigma'}\;, \tag{7}$$

sometimes referred to as the *local quasiparticle energy*. This can be interpreted as the energy of a quasiparticle modified by its interaction with other quasiparticles. From the form of this equation it is clear that, within the Landau theory, this effect is treated in a mean field approximation.

We next turn our attention to the quasiparticle interaction function. An inspection of Eq. (1) reveals that this quantity can be expressed in terms of functional derivatives of the Landau energy functional with respect to the quasiparticle distribution function as

$$f_{\vec{k}\sigma,\vec{k}'\sigma'} = \frac{\delta^2 E}{\delta\mathcal{N}_{\vec{k}\sigma}\delta\mathcal{N}_{\vec{k}'\sigma'}} = \frac{\delta\tilde{\mathcal{E}}_{\vec{k}\sigma}}{\delta\mathcal{N}_{\vec{k}'\sigma'}}\;, \tag{8}$$

where the functional derivatives are evaluated at the ground-state distribution. Notice that in order to correctly perform the second derivative appearing in Eq. (8), one needs to know the energy functional $E[\{\mathcal{N}_{\vec{k}\sigma}\}]$ up to second order in $\delta\mathcal{N}_{\vec{k}\sigma}$. This implies, for instance, that to derive the expression for the interaction function in a paramagnetic system one needs the knowledge of the energy functional appropriate to an infinitesimally polarized electron liquid. This complication does not arise in the case of the quasiparticle energy, since its calculation only requires a knowledge of the Landau energy functional up to first order in $\delta\mathcal{N}_{\vec{k}\sigma}$. Finally, in order to calculate thermal properties at finite temperature T one also needs an expression for the entropy of the liquid. This is given by

$$S\big[\{\mathcal{N}_{\vec{k}\sigma}\}\big] = -k_B\sum_{\vec{k}\sigma}\left(\mathcal{N}_{\vec{k}\sigma}\ln\mathcal{N}_{\vec{k}\sigma} + (1-\mathcal{N}_{\vec{k}\sigma})\ln\left(1-\mathcal{N}_{\vec{k}\sigma}\right)\right)\;, \tag{9}$$

which coincides with the entropy of the noninteracting ensemble of origin, and vanishes in the ground-state. This is a direct consequence of the assumed one-to-one correspondence between states of the interacting and non-interacting systems.

Eqs. (1) and (9) are widely used to calculate, from a macroscopic point of view, the thermal equilibrium properties, the response functions, and the transport properties of an interacting Fermi liquid, and to establish relationships between different such properties. What follows is a summary of the main results.

2.2 The heat capacity

As it turns out, the low-temperature specific heat of a Fermi liquid coincides with that of a noninteracting Fermi gas comprised of particles of mass m^*: it is therefore given by

$$c_v(T) = \frac{\pi^2}{3} N^*(0) L^d k_B^2 T \,, \tag{10}$$

where, as we have seen, $N^*(0)$, the density of quasiparticle energy states per unit volume at the Fermi level, differs from the corresponding quantity in the non interacting case by the substitution of the bare electronic mass m with m^*.[3] This is a direct consequence of Eq. (9) for the entropy, which in turn implies that the quasiparticle distribution function at thermal equilibrium is given by

$$\mathcal{N}^{\rm eq}_{\vec{k}\sigma}(\mu, T) = \frac{1}{e^{\beta(\mathcal{E}_{\vec{k}\sigma} - \mu)} + 1} \,, \tag{11}$$

where $\beta = 1/k_B T$. Notice that the Landau interaction function does not appear in this expression. This is because the thermal excitation of the system does not contribute to the quasiparticle energy. Thus the effective mass can in principle be directly measured from the heat capacity, or by any other measurement that is sensitive only to the quasiparticle density of states. The situation is quite different when the excitation is caused by an external field such as pressure or magnetic field, as we show next.

2.3 The Landau parameters

It is useful to introduce at this point the *Landau Fermi liquid parameters*. One starts from the observation that within the dynamically relevant shell in which $\delta\mathcal{N}_{\vec{k}'\sigma'}$ is finite, the Landau interaction function depends only on the cosine of the angle θ between $\vec{k}$ and $\vec{k}'$. Accordingly we can set $f_{\vec{k}\sigma,\vec{k}'\sigma'} \simeq f_{\sigma\sigma'}(\cos\theta)$ and introduce the dimensionless quantities $F_\ell^{s,a}$ which are defined in terms of spin symmetric (s) and spin antisymmetric (a) angular averages of $f_{\sigma\sigma'}(\cos\theta)$ as follows

$$F_\ell^{s,a} \;=\; \frac{L^d N^*(0)}{2} \int \frac{d\Omega_d}{\Omega_d} \big(f_{\uparrow\uparrow}(\cos\theta) \pm f_{\uparrow\downarrow}(\cos\theta) \big) \begin{cases} P_\ell(\cos\theta) & , \; 3D \\ \cos \ell\theta & , \; 2D \end{cases} \,, \tag{12}$$

where the $+$ and $-$ signs are associated with s and a respectively, $\Omega_d = 2^{d-1}\pi$ is the solid angle in $d = 3$ or 2 dimensions, and $P_\ell(\cos\theta)$ is the ℓ-th Legendre polynomial.

It must be noted that this definition (introduced in Ref. [2]) differs from the one commonly used in previous texts and in large part of the literature. Nervous readers can revert to the standard notation by simply making the substitution $F_\ell^{s,a} \to \frac{F_\ell^{s,a}}{2\ell+1}$ in three dimensions and $F_\ell^{s,a} \to \frac{F_\ell^{s,a}}{2}(1+\delta_{\ell 0})$ in two dimensions.

[3] We are considering here only the mass renormalization that arises from interactions between the particles. In a crystalline environment the crystal potential and the electron-phonon coupling produce additional mass renormalizations.

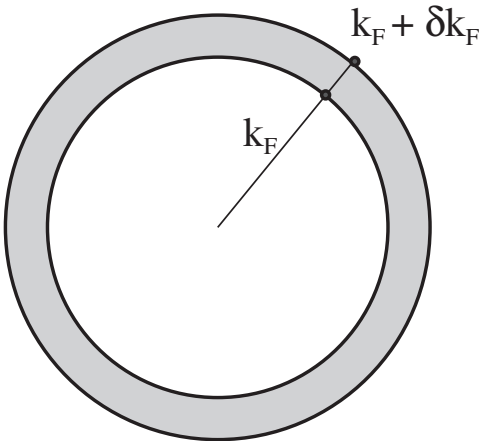

Fig. 2: *Calculation of the compressibility in the Landau theory of Fermi liquids. The chemical potential, i.e., the energy of a quasiparticle at the Fermi surface, changes due to (i) the variation of the Fermi momentum, (ii) the addition of quasiparticles to the shaded region.*

The inverse of Eq. (12) is

$$f_{\uparrow\uparrow}(\cos\theta) \pm f_{\uparrow\downarrow}(\cos\theta) \;=\; \frac{2}{L^d N^*(0)} \sum_{\ell=0}^{\infty} F_\ell^{s,a} \begin{cases} (2\ell{+}1) P_\ell(\cos\theta) & , \; 3D \\ (2{-}\delta_{\ell 0}) \cos\ell\theta & , \; 2D \end{cases} , \tag{13}$$

where $+$ is associated with s and $-$ with a.

2.4 Compressibility and spin susceptibility

An important property of a Fermi liquid is the proper compressibility K,[4] given by the relation

$$\frac{1}{K} \;=\; n^2 \frac{\partial\mu}{\partial n} \;=\; \frac{n k_F}{d} \frac{\partial\mu}{\partial k_F} \, . \tag{14}$$

The compressibility determines, among other things, the magnitude of the screening wave vector and the hydrodynamic sound velocity $s = \frac{1}{\sqrt{nmK}}$. For a non interacting system one simply has the result

$$K_0 \;=\; \frac{N(0)}{n^2} \, . \tag{15}$$

To evaluate the derivative $\frac{\partial\mu}{\partial k_F}$ within the Landau theory of Fermi liquids one must recall that, according to Eq. (3), the change in μ as the Fermi surface expands to accommodate the additional density δn is the sum of two terms: one is the change in the bare quasiparticle energy when the wave vector varies from k_F to $k_F{+}\delta k_F$; the other is the interaction energy with the additional quasiparticles created by the expansion of the Fermi sphere (see Fig. 2). The first term is responsible for changing the density of states in Eq. (15) from $N(0)$ to $N^*(0)$, but the

[4]For a charged Fermi liquid the *proper* compressibility is calculated under the assumption that the system remains charge-neutral during the compression and there is no energy cost associated with the compression of the neutralizing background of charge.

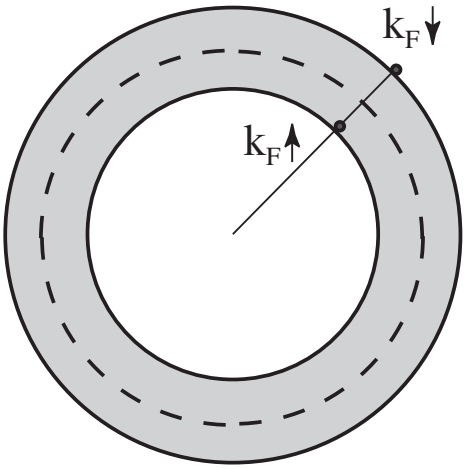

Fig. 3: *Calculation of the spin susceptibility in the Landau theory of Fermi liquids. In the presence of a magnetic field the up- and down-spin Fermi surfaces split in such a way that the energies of two quasiparticles at the two Fermi surfaces are equal. The additional quasiparticles are down-spin electrons above the unperturbed Fermi surface (dashed line) and up-spin holes below it.*

second term involves the Landau parameter F_0^s, since the additional quasiparticle distribution is spherically symmetric ($\ell = 0$) and spin-independent (superscript s). The result is

$$\frac{K}{K_0} = \frac{m^\star/m}{1+F_0^s} \,. \tag{16}$$

Thus, the interaction enters the proper compressibility not only through the effective mass, but also, explicitly, through the spin symmetric spherical average of the Landau interaction function.

The spin susceptibility can be calculated in a completely analogous way (see Fig. 3). In the presence of an external magnetic field B the Hamiltonian is modified by the addition of the Zeeman energy term

$$\hat{H}_Z = \frac{g\mu_B}{2} B \hat{S}_z \,, \tag{17}$$

where g is the *noninteracting* g-factor for the electrons and $\hat{S}_z$ is the z-component of the spin in units of $\frac{\hbar}{2}$. Here $\frac{g\mu_B}{2}$ is the magnetic moment of the electron.[5] The spin susceptibility is defined as the derivative of the magnetization with respect to the magnetic field at zero magnetic field. In an ideal Fermi gas it works out to be

$$\chi_{S0} = \left(\frac{g\mu_B}{2}\right)^2 N(0) \,. \tag{18}$$

In a Fermi liquid the energy of a quasiparticle of wave vector $\vec{k}$ and spin σ in the presence of the magnetic field becomes

$$\mathcal{E}_{\vec{k}\sigma}(B) = \mathcal{E}_{\vec{k}\sigma} + \frac{1}{2} g\mu_B B\sigma \,, \tag{19}$$

where, it must be noted, the g-factor of the quasiparticle coincides with that of the bare electron (as long as spin-orbit interactions are neglected), because the many-body state described by the

[5] For free electrons $g \simeq 2$, but this value can be considerably different for electrons in a solid state environment, due to the spin-orbit interaction: for example in GaAs one has $g = -0.44$.

quasiparticle at $\vec{k}\sigma$ is an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar\sigma/2$. Because of the Zeeman energy, the Fermi surfaces of up-spin and down-spin electrons shift by equal amounts in opposite directions, i.e., the radius of the down-spin Fermi surface increases by an amount $\delta k_{F\downarrow}$, while the radius of the up-spin Fermi surface decreases by the same amount.

The equilibrium value of $\delta k_{F\downarrow}$ is determined by the condition that the energy of an up-spin quasiparticle at the up-spin Fermi surface be equal to that of a down-spin quasiparticle at the down-spin Fermi surface: if this were not the case one could gain energy by transferring quasiparticles from one Fermi surface to the other. The common value of the energy is, of course, the chemical potential (see Eq. (3)). The mathematical form of the equilibrium condition is thus

$$\tilde{\mathcal{E}}_{k_{F\uparrow}\uparrow} + \frac{1}{2}g\mu_B B = \tilde{\mathcal{E}}_{k_{F\downarrow}\downarrow} - \frac{1}{2}g\mu_B B \; . \tag{20}$$

Now, by making use of Eq. (7) and following the same procedure we used for the case of the compressibility, we obtain the elegant result

$$\frac{\chi_S}{\chi_{S0}} \; = \; \frac{m^\star/m}{1+F_0^a} \; , \tag{21}$$

where the interaction enters both through the effective mass and through the Landau parameter F_0^a. This has the same structure as Eq. (16). The $\ell = 0$ component is selected by the spherical symmetry of the quasiparticle distribution, and the a subscript reflects the spin-antisymmetry of that distribution.

Looking at Eqs. (16) and (21) we see that measurements of K and χ_S, combined with a knowledge of the effective mass from the heat capacity allow us to determine the values of the Landau parameters F_0^s and F_0^a. Negative values of these parameters, arising from the exchange interaction enhance both the proper compressibility and the spin susceptibility. Because these two quantities must be finite and positive in a stable ground state, we conclude that the uniform and paramagnetic state will be unstable if F_0^a or F_0^s become less than -1.[6]

2.5 Galilean invariance and effective mass

The effective mass of quasiparticles that we have discussed in the previous section arises entirely from the interaction between the Fermions in a translationally invariant Fermi liquid. Translational invariance is a good assumption for liquid ^{3}He or for nuclear matter, but hardly so for electrons in a solid state environment. The interaction of the electron with the periodic crystal potential and with lattice vibrations is an important source of effective mass renormalization. But if the system is translationally invariant, in the sense that momentum is strictly conserved, and if in addition the kinetic energy is of the Galilean-invariant form $p^2/2m$ with m the bare particle mass, then an exact relation exists between the quasiparticle effective mass and the interaction function:

$$\frac{m^*}{m} = 1 + F_1^s \; , \tag{22}$$

where F_1^s is the $\ell = 1$ (dipolar) component of the spin-symmetric Landau parameter.

[6]For the electron gas only the spin instability is real, as the density instability is preempted by the "improper" contribution arising from the charged background.

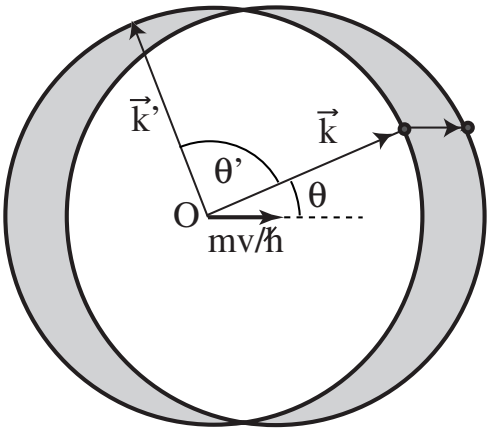

Fig. 4: *Relation between the effective mass and the Landau parameters. When the original Fermi surface centered at O is viewed from a reference frame moving at speed v it appears to be shifted by an amount $mv/\hbar$. The shift can be described in terms of quasiparticles and quasiholes added to the original Fermi distribution in the shaded crescent-shaped regions.*

The origin of this relation is illustrated in Fig. 4. We consider a quasiparticle of momentum $\vec{p} = \hbar\vec{k}$ in the reference frame in which the center of mass of the liquid is at rest. If we now change the reference frame to one in which the center of mass of the liquid moves to the right with velocity $\vec{v}$ the same quasiparticle will appear to have momentum $\hbar\vec{k} + m\vec{v}$ where m is the bare mass of the particle, not the quasiparticle mass! This is because under this transformation the total momentum of the fluid changes by $m\vec{v}$. But the momentum of the quasiparticle is not the only thing that changes. The entire momentum distribution shifts by $m\vec{v}$ and the net result of this shift can be described as the creation of quasiparticles and quasiholes in the shaded crescent-shaped regions of Fig. 4, with quasiparticles residing in the right crescent, and an equal number of quasiholes residing in the left crescent.
The corresponding change in energy of the quasiparticles (calculated to first order in $\vec{v}$) has two contributions:

(i) the change of the single quasiparticle energy $\mathcal{E}_{\vec{k}\sigma}$ due to the fact that the quasiparticle momentum is shifted from $\hbar\vec{k}$ to $\hbar\vec{k} + m\vec{v}$

(ii) the energy of interaction between the quasiparticle of momentum $\hbar\vec{k}$ and the additional quasiparticles and quasiholes that appear in the crescent-shaped regions of Fig. 4

The first contribution involves the effective mass of the quasiparticle, and the second contribution involves the dipolar component of the Landau interaction function (reflecting the dipolar structure of the additional quasiparticle distribution). Combining the two contribution must yield the exact change in the energy of the system under the Galilean transformation, namely $\Delta E = \hbar\vec{v}\cdot\vec{k}$, which does not depend the particle mass. Clearly, this is possible only if a precise relation exists between the quasiparticle mass and the interaction function, and this is what Eq. (22) gives us.
The importance of momentum conservation in the above discussion cannot be overemphasized. Consider, for example, the following question: what is the *spin-current* $j_\uparrow - j_\downarrow$, carried by a

r_s	F_0^s	F_0^a	F_1^s	F_1^a	F_2^s	F_2^a
1	-0.21	-0.17	-0.04	-0.0645	-0.0215	-0.0181
2	-0.37	-0.25	-0.03	-0.0825	-0.0168	-0.0126
3	-0.55	-0.32	-0.02	-0.0915	-0.0107	-0.0073
4	-0.74	-0.37	0.0	-0.0956	-0.0047	-0.0022
5	-0.95	-0.40	0.03	-0.0965	+0.0009	+0.0023

Table 1: *Calculated values of Landau Fermi liquid parameters of the three-dimensional electron liquid. The values of F_1^a, F_2^s, and F_2^a were calculated by Yasuhara and Ousaka [5].*

quasiparticle of wave vector $\vec{k}$ and spin $\uparrow$? Recall that the "spin current" is the difference between the current carried by spin-up particles and that carried by spin-down particles, where the spin of a particle (as well as the spin of a quasiparticle) is a good quantum number. One might be tempted to answer "$\hbar\vec{k}/m$" on the (wrong) assumption that a spin-up quasiparticle carries no down-spin current, but this is incorrect because the difference between the total up- and down-spin momenta of the particles $\hat{\vec{P}}_\uparrow - \hat{\vec{P}}_\downarrow$ is not a constant of the motion. In fact, the magnitude of the *spin-current* is smaller than $\frac{\hbar\vec{k}}{m}$ [4]. What happens is that in the process of switching-on the interaction some momentum is transferred from the up- to the down-spin component of the electron liquid. This reduces the spin current without altering the total momentum and spin. The reduction can be expressed in terms of an effective *spin mass* $m_S > m$ such that $j_\uparrow - j_\downarrow = \frac{\hbar k}{m_S}$. The relation between m_S and m^* has the same form as the relation (22) between the "charge mass", m, and m^*, i.e.,

$$\frac{m^*}{m_S} \;=\; 1 \;+\; F_1^a \;. \tag{23}$$

Numerical values of several Landau parameters of the uniform electron liquid (in the jellium model) obtained from approximate microscopic calculations [5] are listed in Table 2.5 for several different densities.

2.6 Measuring m^*, K and χ_S

The effective mass, the proper compressibility, and the spin susceptibility of liquid ^{3}He have been the object of many experimental studies by a variety of techniques (see, for example, Ref. [6]). Values of the effective mass, the spin susceptibility and the compressibility are reported in Ref. [7]. The effective mass ratio m^*/m ranges between 3 and 6 as the pressure is increased from 0 to 33 bar. In the same interval of pressures the spin susceptibility enhancement factor $(1+F_0^a)^{-1}$ is nearly constant at a value between 3.5 and 4, while the compressibility ratio K/K_0 decreases from 6 to 1 [7].

For electron liquids the situation is generally complicated by the presence of solid-state effects which are hard to identify and eliminate from the analysis. This is especially the case in what would appear to be the best realization of the three-dimensional jellium model, namely elemental metals. In spite of this difficulty valiant attempts have been made to determine a variety of Fermi liquid properties in these systems.

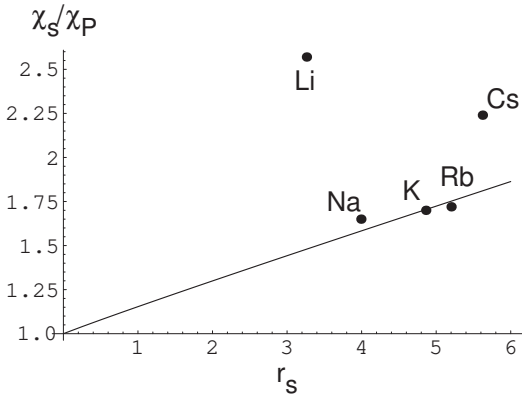

Fig. 5: *Experimentally determined spin susceptibilities of elemental metals vs theoretical values obtained from an analytical fit to quantum Monte Carlo results (see Ref. [2] for details).*

Fig. 5 shows a comparison between the spin susceptibilities of some elemental metals determined via the measurement of the Knight shift in the nuclear magnetic resonance, versus the spin susceptibility calculated directly from the second derivative of the energy with respect to spin polarization, which is known from quantum Monte Carlo calculations. The agreement is quite good for the three middle elements (with intermediate values of r_s) but it is much less satisfactory for the low-density metal Cs, and absolutely disappointing for Li.

The experimental situation improves considerably in two-dimensional electronic systems, which can be made essentially free of lattice and disorder effects. Here the electronic density can be changed by electrical means (gates), thus allowing a systematic study of interaction effects. Thus, for example, measurements of the capacitance of double-layer GaAs quantum wells (Eisenstein, 1994) [8] have allowed a precise determination of the proper compressibility of the two-dimensional electron liquid, confirming, in particular, that this quantity becomes negative below a certain density, without implying an instability of the system (stability being ensured by the presence of remote neutralizing charges on the gates or on the donors).

Coming to the effective mass and the spin susceptibility, the most accurate measurements so far have been done on systems such as n-type Silicon inversion layers, n-doped GaAs/AlGaAs, and p-doped GaAs (see Ref. [2] for details). While the early experiments were done at densities of the order of 10^{12} cm^{-2}, corresponding to $r_s \sim 3$ in Si, the most recent ones have reached considerably lower densities in both GaAs (1.7×10^9 cm^{-2}, i.e. $r_s = 13.4$) and in Si inversion layers (10^{11} cm^{-2}, i.e. $r_s = 8.4$). At such low densities, the many-body renormalizations (i.e., the Landau parameters) are strong: for example, the spin susceptibility enhancement $\frac{m^* g^*}{m_b g_b}$ (where m_b and g_b are the "bare" mass and g-factor determined by the band structure of the host semiconductors: $g_b \sim 2$ in Si and $g_b \sim 0.44$ in GaAs) can be as large as 5. [9] This is consistent with the theoretical expectation that the 2DEG should become ferromagnetic at sufficiently low density: however, no ferromagnetic instability has been observed so far.

Recently, tilted field experiments have allowed the determination of the spin susceptibility of the two-dimensional electron liquid in very narrow AlAs quantum wells in a broad range of densities [10]. Fig. (6) shows the comparison, obviously very satisfactory, between the measured data and the values obtained from quantum Monte Carlo calculations.

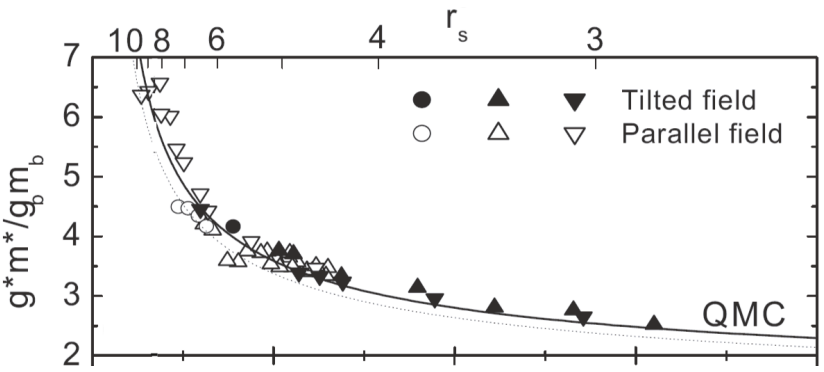

Fig. 6: *Spin susceptibility enhancement vs r_s for the 2DEL in narrow AlAs quantum wells. Circles and triangles denote experimental data taken in different samples and for different orientations of the magnetic field. The quantum Monte Carlo predictions at zero and full spin polarization are shown by the dotted and solid lines respectively. From Vakili et al. [10].*

3 The lifetime of quasiparticles

3.1 General formulas

Up to this point we have treated quasiparticles as if they were exact eigenstates of the many-fermion system. In reality, as we pointed out in the introduction, quasiparticles are not exact eigenstates, and do decay in time by transferring their momentum and energy to other quasiparticles. The key point of the Landau theory of Fermi liquids is that the decay rate tends to zero much more rapidly than the quasiparticle energy (relative to the chemical potential) when the latter tends to zero. It is this circumstance that allows us in a first approximation to neglect the decay rate and to treat the interaction between quasiparticles at the mean-field level, i.e., simply as a modification of the quasiparticle energy which is describable in terms of the Landau interaction function. However, the residual interactions, beyond the mean-field approximation, cause quasiparticles to decay. This decay process plays an essential role in the study of transport phenomena, where an external field drives the quasiparticles out of their thermal equilibrium distribution. In this situation the residual interactions between quasiparticles pull the quasiparticle distribution back towards equilibrium, thus providing a kind of restoring force which determines the magnitude of the response to the external field.

The most relevant process contributing to the decay of a quasiparticle state in a Fermi liquid is particle-hole pair production. At zero temperature, given a quasiparticle close to the Fermi surface, there is a certain probability that, because of the interaction, part of the energy and momentum of the quasiparticle will be lost by exciting a single electron-hole pair out of the Fermi sea. Upon losing part of its initial momentum and energy the quasiparticle makes a transition to an available lower energy state. At finite temperatures the scenario is slightly complicated by the fact that the available final states are neither definitely occupied nor definitely empty, but the basic physical picture remains the same.

In principle a quasiparticle can also lose momentum and energy by exciting *multiple* particle-hole pairs and/or collective modes (zero sound in a neutral Fermi liquid or plasmons in the electron liquid). As it turns out both processes are irrelevant for quasiparticles near the Fermi

surface at sufficiently low temperature. Decay by emission of collective modes is forbidden by energy and momentum conservation, since collective modes owe their very existence to having frequencies that are significantly higher than particle-hole excitation energies at the same momentum. As for multiple particle-hole pair excitations, we note that the spectral density of these excitations vanishes, at low energy, much more rapidly than the spectral density of single particle-hole pairs.

The simplest way to estimate the particle-hole contribution to the quasiparticle decay rate is to make use of the *Fermi golden rule* to compute the transition probability between the initial and the final state of the system. Within this approach the rate at which a quasiparticle of spin σ and momentum $\hbar\vec{k}$ is scattered by the Coulomb interaction into an empty state of momentum $\hbar(\vec{k}-\vec{q})$, while a second quasiparticle of spin σ' and momentum $\hbar k'$ is scattered into an empty state of momentum $\hbar(\vec{k}'+\vec{q})$ is given by

$$\frac{2\pi}{\hbar}\left|\frac{W(\vec{q})}{L^d}\right|^2 \delta\big(\varepsilon_{\vec{k}-\vec{q}\sigma}+\varepsilon_{\vec{k}'+\vec{q}\sigma'}-\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}'\sigma'}\big), \tag{24}$$

where $\frac{W(\vec{q})}{L^d}$ is the matrix element of an effective interaction between quasiparticles, taken between the initial and final plane-wave states. The δ-function ensures that the energy is conserved through the collision.

Eq. (24) is approximate in more than one way. First of all, the correct two-particle scattering amplitude is a function of $\vec{k}, \vec{k}'$, and $\vec{q}$, as well as the relative spin orientation, not just of the momentum transfer $\vec{q}$. Secondly, Eq. (24) violates the indistinguishability of the particles, since the scattering amplitude is not antisymmetric upon interchange of the two final plane-wave states of parallel spin. Finally, Eq. (24) is not entirely self-consistent, since the quasiparticle energy is approximated by the bare particle energy. In spite of these defects, Eq. (24) is still an excellent starting point to begin to understand the microscopic physics of the Fermi liquid. In what follows I focus on the case of the electron liquid, where the long range of the Coulomb interaction requires some special attention: however the final result has the same form for all Fermi liquids.

The inverse lifetime $1/\tau^{(e)}_{\vec{k}}$ of a plane wave state initially occupied by an electron of momentum $\hbar\vec{k}$ and spin σ, is given by the sum of the probabilities of all the allowed decay processes:

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} = \frac{2\pi}{\hbar}\sum_{\vec{q}\vec{k}'\sigma'}\left|\frac{W(\vec{q})}{L^d}\right|^2 n_{\vec{k}'\sigma'}\big(1-n_{\vec{k}'+\vec{q}\sigma'}\big)\big(1-n_{\vec{k}-\vec{q}\sigma}\big)\,\delta\big(\varepsilon_{\vec{k}-\vec{q}\sigma}+\varepsilon_{\vec{k}'+\vec{q}\sigma'}-\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}'\sigma'}\big), \tag{25}$$

where the Fermi occupation factors guarantee that the plane-wave state $\vec{k}'\sigma'$ is indeed occupied by an electron, while the final states $\vec{k}'+\vec{q}\,\sigma'$, and $\vec{k}-\vec{q}\,\sigma'$ are empty and therefore available for occupation after the scattering event.

A nice feature of Eq. (25) is that part of the calculation can be carried out without specifying the form of the scattering amplitude $W(\vec{q})$. We work, for simplicity, in the paramagnetic state, and approximate $n_{\vec{k}\sigma}$ by the *noninteracting* occupation numbers $n^{(0)}_{\vec{k}\sigma}$. Then we make use of the fluctuation-dissipation theorem for the non-interacting electron gas (see Ref. [2] for details) to

evaluate the sum over $\vec{k}'$ and σ':

$$\frac{\pi}{\hbar L^d}\sum_{\vec{k}'\sigma'} n^{(0)}_{\vec{k}'\sigma'}\left(1-n^{(0)}_{\vec{k}'+\vec{q}\,\sigma'}\right)\delta\left(\frac{\varepsilon_{\vec{k}'+\vec{q}\,\sigma'}-\varepsilon_{\vec{k}'\sigma'}}{\hbar}-\omega\right) \;=\; -\frac{\mathrm{Im}\,\chi_0(q,\omega)}{1-e^{-\beta\hbar\omega}}\,. \tag{26}$$

Here $\chi_0(q,\omega)$ is the density-density response function of the noninteracting Fermi liquid, also known as the Lindhard function [2]. Naturally, the appearance of the spectral density of particle-hole pairs, $-\,\mathrm{Im}\,\chi_0(q,\omega)$, shows that these excitations play a central role in the process. Eq. (25) can now be rewritten as

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} = -\frac{2}{(2\pi)^d}\int_{-\infty}^{\infty}d\omega\frac{1-n_F(\varepsilon_{\vec{k}\sigma}-\hbar\omega-\mu)}{1-e^{-\beta\hbar\omega}}\int_0^{\infty}dq q^{d-1}\left|W(\vec{q})\right|^2\mathrm{Im}\,\chi_0(q,\omega)\int d\Omega_d\delta\left(\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}-\vec{q}\sigma}-\hbar\omega\right) \tag{27}$$

where $n_F(x)=1/(e^{\beta x}+1)$ is the Fermi-Dirac distribution function at zero chemical potential, so that $n_F(\varepsilon_{\vec{k}\sigma}-\mu)=1/(e^{\beta(\varepsilon_{\vec{k}\sigma}-\mu)}+1)=n^{(0)}_{\vec{k}\sigma}$ is the noninteracting occupation number. In obtaining this expression we have introduced the variable $\hbar\omega=\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}-\vec{q}\sigma}$ through the introduction of an auxiliary delta function and its corresponding integration. Notice that the angular integration only involves the delta function.

The corresponding formula for the lifetime of a quasihole ($\tau^{(h)}_{\vec{k}\sigma}$) is obtained from Eq. (27) by performing the replacements $1-n_F(\varepsilon_{\vec{k}\sigma}-\hbar\omega-\mu)\to n_F(\varepsilon_{\vec{k}\sigma}-\hbar\omega-\mu)$ and $1-e^{-\beta\hbar\omega}\to 1-e^{\beta\hbar\omega}$, and changing the overall sign. This can be used to demonstrate that the principle of detailed balance (see Eq. (50) below), is satisfied at the appropriate level of accuracy, i.e., we have

$$\frac{n_{\vec{k}\sigma}}{\tau^{(e)}_{\vec{k}\sigma}} \;=\; \frac{1-n_{\vec{k}\sigma}}{\tau^{(h)}_{\vec{k}\sigma}}\,. \tag{28}$$

This important result guarantees (see Section 3.5) that the equilibrium Fermi-Dirac distribution of quasiparticles is stable against quasiparticle decay processes.

We shall henceforth concentrate only on the calculation of the quasiparticle lifetime. The calculation will be carried out separately for three and two-dimensional systems below. In one dimension, a calculation of $1/\tau^{(e)}_{\vec{k}\sigma}$ based on Eq. (27) would result in a divergent integral at finite temperatures: we can conclude that the Landau Fermi liquid picture cannot be consistently applied to 1D systems, as mentioned in the introduction.

3.2 Three-dimensional electron gas

In this case the angular integration is rather simple once the z-axis is taken along the direction of $\vec{q}$. We have:

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\,\delta\left(\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}-\vec{q}\sigma}-\hbar\omega\right)=\frac{2\pi m}{\hbar^2 kq}\,\Theta\left(1-\left|\frac{q^2+\frac{2m\omega}{\hbar}}{2kq}\right|\right), \tag{29}$$

where $\Theta(x)$ is the familiar Heaviside step function which mandates a precise behavior for the limits of the remaining quadratures. As it turns out however, these limiting conditions are

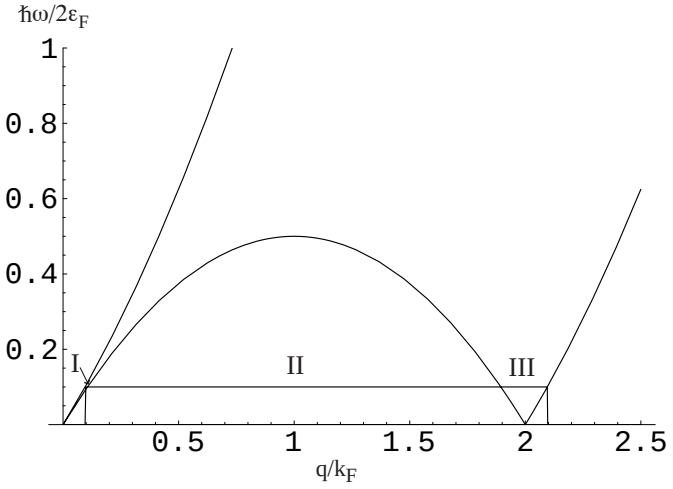

Fig. 7: *Regions of wave vector integration for the calculation of the low-energy quasiparticle lifetime. For $k \to k_F$ and $\hbar\omega \ll \epsilon_F$ the leading contribution to the wave vector integral comes from region II where $\frac{|\omega|}{v_F} < q < 2k_F - \frac{|\omega|}{v_F}$. Two-dimensional plasmon excitations do not contribute to the lifetime and hence are not shown.*

irrelevant, in view of the behavior of the integrand. This can be seen as follows: for positive frequencies the Fermi thermal occupation factor $1 - n_F(\varepsilon_{\vec{k}\sigma} - \hbar\omega - \mu)$ cuts off the integral for ω of the order of $|\varepsilon_{\vec{k}\sigma} - \mu|$, an energy which, by definition, is much smaller than the Fermi energy ϵ_F. For negative frequencies, on the other hand, it is the thermal occupation factor $1/(1 - e^{-\beta\hbar\omega})$ that cuts off the frequency integral for ω of the order of $k_B T$, an energy scale that we assume to be much smaller than ϵ_F. At very low frequency the most stringent limits on the q-integral come from the factor $\mathrm{Im}\,\chi_0(q,\omega)$ which contains the structure of the particle-hole continuum and differs from zero only along the segment shown in Fig. 7. This sets the lower limit of the q integral at $q \sim \frac{|\omega|}{v_F} \sim 0$ and the upper limit at $q \sim 2k_F + \frac{|\omega|}{v_F} \sim 2k_F$. Notice that the dominant contribution to the integral (for $\omega \to 0$) comes from the region labeled as II in Figure 7. In this region it is legitimate to approximate $\mathrm{Im}\,\chi_0(q,\omega)$ by its zero temperature and low-frequency form

$$\mathrm{Im}\,\chi_0(q,\omega) \simeq -\frac{\pi\omega}{2v_F q} N(0)\,. \tag{30}$$

Accordingly the formula for $1/\tau_k^{(e)}$ in three dimensions is seen to be proportional to

$$\frac{1}{\tau_{\vec{k}\sigma}^{(e)}} \;\propto\; \int_{-\infty}^{\infty} \frac{d\omega\;\omega}{\left(1 + e^{\beta(\hbar\omega - \varepsilon_{\vec{k}\sigma} + \mu)}\right)\left(1 - e^{-\beta\hbar\omega}\right)} \int_0^{2k_F} dq\,\left|W(\vec{q})\right|^2 . \tag{31}$$

The frequency integral can be calculated analytically and is given by

$$\frac{1}{2\hbar^2} \frac{(\varepsilon_{\vec{k}\sigma} - \mu)^2 + (\pi k_B T)^2}{1 + e^{-\beta(\varepsilon_{\vec{k}\sigma} - \mu)}}\,. \tag{32}$$

The integral over the wave vector q deserves special attention. It is quite obvious at this point that one cannot make use of the bare Coulomb interaction. The integral would emphatically diverge. This is of course a consequence of the long range of the Coulomb interaction. It is then

quite natural to employ for $W(\vec{q})$ some sort of screened interaction. In this case the q integral presents no problems. A reasonable approximation is provided by the choice $W(\vec{q}) \simeq \frac{v_q}{\varepsilon(q,0)}$ so that

$$\int_0^{2k_F} dq \left|W(\vec{q})\right|^2 \simeq \frac{1}{N^2(0)} \int_0^{2k_F} dq \left(\frac{N(0)v_q}{\varepsilon(q,0)}\right)^2 . \tag{33}$$

Further simplification can be achieved by making use of the Thomas-Fermi approximate dielectric function, that is the static long wavelength limit of the RPA dielectric function, given by $\epsilon(q,0) = 1 + \frac{4\pi e^2 N(0)}{q^2}$. This gives the result

$$\int_0^{2k_F} dq \left|W(\vec{q})\right|^2 \simeq \frac{2k_F}{N^2(0)} \xi_3(r_s) , \tag{34}$$

where the function $\xi_3(r_s)$ is given by

$$\xi_3(r_s) = \sqrt{\frac{\alpha_3 r_s}{4\pi}} \tan^{-1} \sqrt{\frac{\pi}{\alpha_3 r_s}} + \frac{1}{2\left(1+\frac{\pi}{\alpha_3 r_s}\right)} . \tag{35}$$

Recall that $r_s (= \left(\frac{3}{4\pi n a^3}\right)^{1/3}$ in 3D) is the average distance between electrons in units of the Bohr radius $a = \frac{\hbar^2}{me^2}$. For most densities in the metallic range $\xi_3(r_s) \simeq 1$. Notice that $\xi_3(r_s) \sim \sqrt{r_s}$ as $r_s \to 0$: thus, due to the non-perturbative nature of the screening, the quasiparticle decay rate turns out being proportional to the electron charge e, rather than e^4, as one could have naively expected.

Collecting the various factors we finally obtain for the inelastic quasiparticle lifetime in three dimensions the following result

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} \simeq \frac{\pi}{8\hbar\epsilon_F} \frac{(\varepsilon_{\vec{k}\sigma}-\epsilon_F)^2 + (\pi k_B T)^2}{1 + e^{-\beta(\varepsilon_{\vec{k}\sigma}-\epsilon_F)}} \xi_3(r_s) \qquad \text{in } 3D, \tag{36}$$

where we have approximated μ with ϵ_F and k with k_F.

The inverse lifetime of a quasiparticle at the Fermi surface ($k = k_F$) vanishes as T^2 at small temperatures. On the other hand, at $T = 0$, the inverse lifetime vanishes as $(\varepsilon_{\vec{k}\sigma}-\epsilon_F)^2$. One power of $\varepsilon_{\vec{k}\sigma}-\epsilon_F$ (or T) arises from the phase space restrictions on the scattering process. The second one stems from the linearly vanishing density of particle-hole pair excitations. The numerical prefactor is simply a Fermi surface average of the statically screened Coulomb interaction. This is the expected behavior, an indication that the Landau theory of the electron liquid is consistent with the microscopic perturbative approach.

3.3 Two-dimensional electron gas

The two-dimensional case presents a few new twists. The most important difference to the three-dimensional case is the q dependence of the integrand of Eq. (27), which must be handled with special care in the regions $q \simeq 0$ and $q \simeq 2k_F$. This necessitates a more precise treatment of the limits of integration.

We begin by considering the angular integration which in this case gives the interesting result

$$\int_0^{2\pi} d\phi\, \delta\big(\varepsilon_{\vec{k}\sigma}-\varepsilon_{\vec{k}-\vec{q}\sigma}-\hbar\omega\big) \;=\; \frac{2\Theta\left(1-\left|\frac{q^2+\frac{2m\omega}{\hbar}}{2kq}\right|\right)}{\sqrt{\left(\frac{\hbar^2 kq}{m}\right)^2-\left(\hbar\omega+\frac{\hbar^2q^2}{2m}\right)^2}}\,, \tag{37}$$

an expression that features an extra frequency dependence with respect to the three-dimensional case. The other necessary ingredient is the expression for $\mathrm{Im}\,\chi_0(q,\omega)$ in two dimensions, which, at low frequency and in region II of Fig. 7, is approximately given by

$$\mathrm{Im}\,\chi_0(q,\omega) \simeq -\frac{\omega}{qv_F}\frac{N(0)}{\sqrt{1-\left(\frac{q}{2k_F}\right)^2}} \qquad (2D), \tag{38}$$

where $N(0)=\frac{m}{\pi\hbar^2}$. Within the necessary accuracy, we can set $k=k_F$ in the argument of the square root appearing in Eq. (37), which can then be rewritten as

$$\frac{\hbar^2}{2m}\sqrt{\left(q^2-\left(\frac{|\omega|}{v_F}\right)^2\right)\left(4k_F^2-q^2\right)}\,. \tag{39}$$

Accordingly we see that the contribution of region II to the q integral of Eq. (27) is

$$8\pi k_F\int_{\frac{|\omega|}{v_F}}^{2k_F-\frac{|\omega|}{v_F}}\frac{\big|N(0)W(\vec{q})\big|^2 dq}{\sqrt{q^2-\left(\frac{|\omega|}{v_F}\right)^2}\,(4k_F^2-q^2)} \qquad (2D). \tag{40}$$

This integral can be evaluated rather easily. Notice that in the limit $\omega\to 0$ it presents a logarithmic divergence originating from the regions $q\simeq 0$ and $q\simeq 2k_F$. To extract the exact coefficient of the logarithmic singularity we set $q=0$ and $q=2k_F$ in the regular parts of the integrand, when evaluating the contributions of $q\simeq 0$ and $q\simeq 2k_F$ respectively. Up to corrections that remain finite as $\omega\to 0$ the integral is then found to be equal to

$$\frac{\pi\big(|N(0)W(0)|^2+|N(0)W(2k_F)|^2/2\big)}{k_F}\,\ln\frac{4\epsilon_F}{|\omega|}\,. \tag{41}$$

It is convenient, at this point, to define the "coupling constant"

$$\xi_2(r_s)\equiv|N(0)W(0)|^2+\frac{1}{2}|N(0)W(2k_F)|^2\,, \tag{42}$$

which, in the Thomas-Fermi approximation (see [2]), depends on r_s in the following manner:

$$\xi_2(r_s)=1+\frac{1}{2}\left(\frac{r_s}{r_s+\sqrt{2}}\right)^2\,. \tag{43}$$

Notice that, unlike its three-dimensional counterpart, $\xi_2(r_s)$ tends to a constant, 1, in the high-density limit.[7] Combining Eqs. (27), (37), (38), and (40), we find that the quasiparticle lifetime

[7] The surprising fact that the inverse lifetime fails to vanish in the noninteracting limit $r_s\to 0$ is an artifact due to our asymptotic expansion of the integral (40), which requires the limit $k\to k_F$ to be taken *before* the limit $r_s\to 0$. The expansion fails for $r_s<|k/k_F-1|$.

is given by the integral

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} \simeq \frac{\hbar\xi_2(r_s)}{2\pi\epsilon_F}\int_{-\infty}^{\infty}\frac{\omega\ln\frac{4\epsilon_F}{|\omega|}}{\left(1+e^{\beta(\hbar\omega-\varepsilon_{\vec{k}\sigma}+\mu)}\right)\left(1-e^{-\beta\hbar\omega}\right)}\,d\omega \qquad \text{in } 2D. \tag{44}$$

Consider first the "zero-temperature" situation $|\varepsilon_{\vec{k}\sigma}-\epsilon_F| \gg k_BT$. In this case it is clear that the main contribution to the integral comes from the region $\omega \simeq \varepsilon_{\vec{k}\sigma}-\epsilon_F$. Now, since in this region the logarithm is slowly varying, we can take it out of the integration to give the factor $\ln\frac{4\epsilon_F}{|\varepsilon_{\vec{k}\sigma}-\epsilon_F|}$. This leaves us with a frequency integral which coincides with that of Eq. (31), which we calculated exactly. The only difference is that in this case we need to take the limit $\frac{k_BT}{|\varepsilon_{\vec{k}\sigma}-\epsilon_F|} \to 0$. By making use of Eq. (32) we therefore obtain the result

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} \simeq \xi_2(r_s)\frac{(\varepsilon_{\vec{k}\sigma}-\epsilon_F)^2}{4\pi\hbar\epsilon_F}\ln\frac{4\epsilon_F}{|\varepsilon_{\vec{k}\sigma}-\epsilon_F|}\,, \quad k_BT \ll |\varepsilon_{\vec{k}\sigma}-\epsilon_F|\,. \tag{45}$$

The other relevant case is that of $k_BT \gg |\varepsilon_{\vec{k}\sigma}-\epsilon_F|$, which corresponds to the case of a quasiparticle lying on the Fermi surface. In this case a direct inspection of Eq. (44) shows that the most relevant contributions to the integral come from a region of the order of $k_BT/\hbar$ centered about the origin. In this situation the logarithm can again be taken out of the integral[8] as to give a factor of $\ln\frac{4\epsilon_F}{k_BT}$. The remaining integral can then again be evaluated by means of Eq. (32) in the limit of $\frac{|\varepsilon_{\vec{k}\sigma}-\epsilon_F|}{k_BT} \to 0$. This immediately gives[9]

$$\frac{1}{\tau^{(e)}_{\vec{k}\sigma}} \simeq \xi_2(r_s)\frac{(\pi k_BT)^2}{8\pi\hbar\epsilon_F}\ln\frac{4\epsilon_F}{k_BT}\,, \quad |\varepsilon_{\vec{k}\sigma}-\epsilon_F| \ll k_BT\,. \tag{46}$$

The only significant difference with the three-dimensional case is the appearance here of the logarithmic factors. This fact was first discovered by Giuliani and Quinn in Ref. [11].
The basic comments made about the three-dimensional result continue to apply. We emphasize that the above calculation focused on the leading term in the low-energy/low-temperature expansion of the inverse lifetime.[10] The complete calculation of the "subleading" contributions of order $(\varepsilon_{\vec{k}\sigma}-\epsilon_F)^2$ and $(k_BT)^2$ is more tricky: in particular, it can be shown that the regions I and III in the q integral (see Fig. 7) do contribute to the result at this order.

3.4 Measuring the quasiparticle lifetime

Thanks to the great improvements in the manufacture of high-quality quantum well systems it has become possible to directly measure, by means of precise and elegant tunneling experiments between parallel identical quantum wells, the quasiparticle lifetime of a two-dimensional electron liquid (Murphy *et al.*, Ref. [12]).

[8] This is due to the fact that if $f(x)$ is a well behaved function in the interval $[-a,a]$, then in the limit $a \to 0$, with logarithmic accuracy, $\int_{-a}^{a}\ln|x|f(x)\,dx \simeq \ln|a|\int_{-a}^{a}f(x)\,dx$ as one can readily verify.

[9] The coefficient of Eq. (46) can also be quickly inferred from Eq. (45) by making use of the general result (32).

[10] It is somewhat sobering to remark that a surprisingly vast variety of contradicting results for the coefficients of Eqs. (45) and (46) have appeared in the literature.

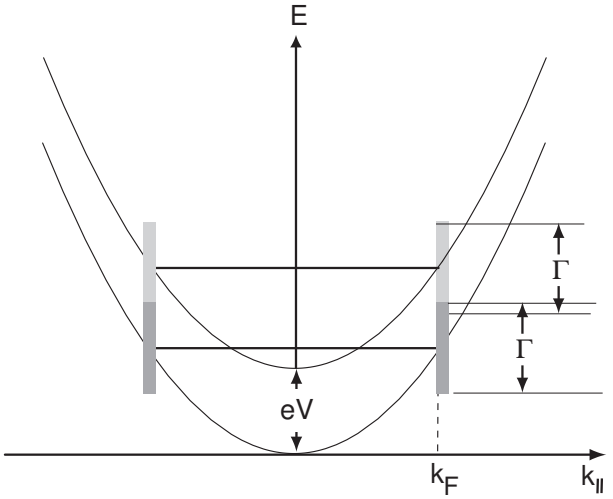

Fig. 8: *Momentum-conserving tunneling between two identical free-electron bands separated by a potential difference eV. Because of energy conservation, the tunneling probability decreases rapidly when eV exceeds the spectral width Γ of the single-particle states in each band.*

The basic idea of the experiment is shown in Fig. 8. The two identical parabolas separated by an energy eV represent the energy vs wave vector relation of the quasiparticle states in the two quantum wells, and the shaded regions show the energy spread of these states due to finite lifetime of a quasiparticle at the Fermi surface (we are, of course, at finite temperature). Here $k_\parallel$ is the two-dimensional in-plane-wave vector, V is the electric potential difference between the two quantum wells, and Γ is the width of the quasiparticle peak in the spectral function at $k_\parallel = k_F$, which is directly related to the quasiparticle lifetime. Under the assumption that electron-impurity and electron-phonon scattering are negligible the two-dimensional wave vector of the tunneling electrons is conserved and overall energy conservation causes the tunneling probability to decrease sharply when eV exceeds Γ. More precisely, a plot of the tunneling conductance vs voltage is approximately a Lorentzian centered at zero voltage with width at half maximum equal to 2Γ, as shown in the inset of Fig. 9. From this width the quasiparticle lifetime can be inferred.
In practice, the interpretation of the experimental data is complicated by the presence of disorder, which leads to imperfect momentum conservation and a finite linewidth even in the limit of zero temperature. This extrinsic contribution, however, is expected to be nearly independent of temperature, and when one subtracts it from the data one obtains values that are in reasonably good agreement with the theory presented in this section (see Fig. 9).

3.5 The kinetic equation

Perhaps the technical centerpiece of the Landau theory of Fermi liquids is the *kinetic equation*, which governs the time evolution of the quasiparticle distribution function in out-of-equilibrium situations, such as in the presence of slowly varying external fields. The main idea is to treat the system as an assembly of quasiparticle wave packets characterized by an average position $\vec{r}$ and an average momentum $\hbar\vec{k}$ ($k \simeq k_F$). The quantum mechanical uncertainties in position and energy are assumed to be negligible on the scale of spatial and temporal variation of the external

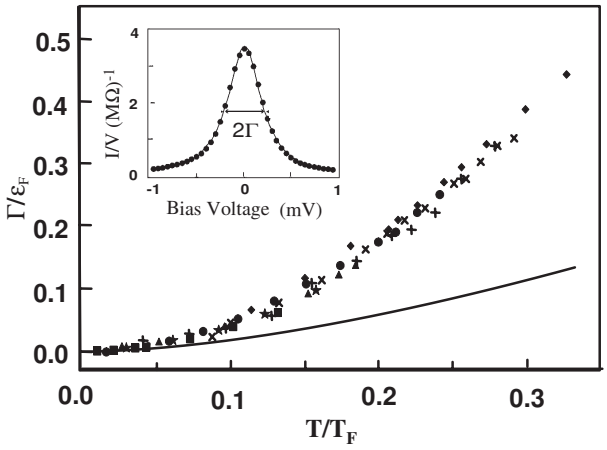

Fig. 9: *Inset: Lorentzian lineshape of the current-voltage (I-V) relation in tunneling between two-dimensional GaAs quantum wells. Main figure: a plot of the half-width Γ, identified with the inverse of the quasiparticle lifetime, vs temperature for systems of different density (a residual T=0 contribution, attributed to disorder, has been subtracted). The solid line is the theoretical prediction from Eq. (46) which is only applicable asymptotically. Adapted from Ref. [12].*

fields. This description makes sense only if the wavevector and frequency of the external field are much smaller than the Fermi wave vector and the Fermi energy respectively. In addition, the thermal energy k_BT must be much smaller than the Fermi energy in order that the notions of Fermi surface and quasiparticles be well defined. Under these assumptions the quasiparticle wave packets can be treated as classical particles, with canonical coordinates and momenta $\vec{r}$ and $\hbar\vec{k}$, described by a "classical" Hamiltonian

$$H_{cl}(\vec{r},\hbar\vec{k},\sigma) = \mathcal{E}_{\vec{k}\sigma} - e\phi_\sigma(\vec{r},t) + \sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta\mathcal{N}_{\vec{k}'\sigma'}(\vec{r},t) \tag{47}$$

where $\phi_\sigma(\vec{r},t)$ is a (generally spin-dependent) scalar potential. The last term on the right hand side of Eq. (47) describes the effect of the *short-range* interaction between the quasiparticles. It has the form of a mean effective potential whose strength is controlled by the Landau interaction function. The long-range electrostatic potential (Hartree potential) is self-consistently included in the external field.

The equation of motion for the quasiparticle distribution function follows immediately from Liouville's theorem for a classical flow in phase space

$$\frac{\partial\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial t} + \frac{1}{\hbar}\frac{\partial H_{cl}}{\partial\vec{k}}\cdot\frac{\partial\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial\vec{r}} - \frac{1}{\hbar}\frac{\partial H_{cl}}{\partial\vec{r}}\cdot\frac{\partial\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial\vec{k}} = \left(\frac{\partial\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial t}\right)_{\rm coll}. \tag{48}$$

The collisional time derivative on the right hand side of Eq. (48) takes into account the fact that the evolution of the quasiparticle distribution function is affected by collision processes that are not included in the classical mean field Hamiltonian. As discussed in previous sections, quasiparticle collisions result in a finite lifetime of quasiparticles ($\tau^{(e)}_{\vec{k}\sigma}$) and quasiholes ($\tau^{(h)}_{\vec{k}\sigma}$) near the Fermi surface. We can therefore write

$$\left(\frac{\partial\mathcal{N}_{\vec{k}\sigma}}{\partial t}\right)_{\rm coll} = -\frac{\mathcal{N}_{\vec{k}\sigma}}{\tau^{(e)}_{\vec{k}\sigma}} + \frac{1-\mathcal{N}_{\vec{k}\sigma}}{\tau^{(h)}_{\vec{k}\sigma}}\,, \tag{49}$$

where the first term represents the rate at which quasiparticles leave the state $\vec{k}\sigma$ and the second is the rate at which they are scattered into it. Besides the interaction contributions derived in Section 3, the total decay rates will in general include contributions from electron-impurity and electron-phonon scattering.

It is evident that the collisional derivative must vanish when $\mathcal{N}_{\vec{k}\sigma}$ is the thermal equilibrium distribution $\mathcal{N}^{\rm eq}_{\vec{k}\sigma}$, i.e., the Fermi-Dirac distribution with energy $\mathcal{E}_{\vec{k}\sigma}$ (see Eq. (11)). This *principle of detailed balance* leads to an exact relation between quasiparticle and quasihole lifetimes:

$$\frac{\mathcal{N}^{\rm eq}_{\vec{k}\sigma}}{\tau^{(e)}_{\vec{k}\sigma}} \;=\; \frac{1-\mathcal{N}^{\rm eq}_{\vec{k}\sigma}}{\tau^{(h)}_{\vec{k}\sigma}}\,, \tag{50}$$

which is satisfied (at the appropriate level of accuracy) by the formulas presented in Section 3 (see Eq. (28)). Expanding Eq. (48) to first order in the strength of the external fields we obtain the *linearized kinetic equation* for the deviation of the distribution function from equilibrium. This equation has the form

$$\frac{\partial\delta\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial t} + \vec{v}_{\vec{k}\sigma}\cdot\frac{\partial\delta\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial\vec{r}} + \vec{v}_{\vec{k}\sigma}\cdot\vec{\mathcal{F}}_{\vec{k}\sigma}(\vec{r},t)\,\delta(\mathcal{E}_{\vec{k}\sigma}-\mu) = \left(\frac{\partial\delta\mathcal{N}_{\vec{k}\sigma}(\vec{r},t)}{\partial t}\right)_{\rm coll} \tag{51}$$

where $v_{\vec{k}\sigma} = \hbar\vec{k}/m^*$ is the quasiparticle velocity, and

$$\vec{\mathcal{F}}_{\vec{k}\sigma}(\vec{r},t) = -\vec{\nabla}_{\vec{r}}\left(-e\phi_\sigma(\vec{r},t) + \sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta\mathcal{N}_{\vec{k}'\sigma'}(\vec{r},t)\right) \tag{52}$$

is the classical force acting on the quasiparticle. This equation is the starting point for most applications of the Landau theory of Fermi liquids.

One outstanding application of the kinetic equation is the study of the macroscopic dynamics of the quasiparticle distribution function in the absence of external fields, leading to the prediction of self-sustained collective modes (i.e., normal modes of oscillation of the Fermi surface) of different symmetries. In this manner one can obtain the (long-wavelength) dispersion of the zero-sound mode in the neutral Fermi liquid and plasmons in the electron liquid. Focusing, for example, on plasmons, we neglect the collision term (justified, since the collision rate is much smaller than the plasmon frequency) and notice that $\phi_\sigma(\vec{r},t)$ is the self-consistent electrostatic potential (Hartree potential) associated with a density fluctuation $\delta n(\vec{r},t) = \sum_{\vec{k}'\sigma'}\delta\mathcal{N}_{\vec{k}'\sigma'}(\vec{r},t)$. We take advantage of linearity by performing a Fourier transformation with respect to the variables $\vec{r}$ and t in Eqs. (51) and (52). This gives us

$$\big(\omega-\vec{q}\cdot\vec{v}_{\vec{k}\sigma}\big)\delta\mathcal{N}_{\vec{k}\sigma}(\vec{q},\omega) + \vec{q}\cdot\vec{v}_{\vec{k}\sigma}\,\delta\big(\mathcal{E}_{\vec{k}\sigma}-\mu\big)\sum_{\vec{k}'\sigma'}\big(v_q+f_{\vec{k}\sigma,\vec{k}'\sigma'}\big)\delta\mathcal{N}_{\vec{k}'\sigma'}(\vec{q},\omega) = 0\,, \tag{53}$$

where $v_q = 4\pi e^2/q^2$ in 3D and $v_q = 2\pi e^2/q$ in 2D. A nontrivial solution $\delta\mathcal{N}_{\vec{k}'\sigma'}(\vec{q},\omega) \neq 0$ of this equation exists only if ω equals the plasmon frequency $\omega_p(q)$. In the long wavelength limit, $q\to 0$, we get $\omega_p(q) = (4\pi ne^2/m)^{1/2}$ in 3D and $\omega_p(q) = (2\pi ne^2 q/m)^{1/2}$ in 2D. Notice that these results involve the bare electron mass m, not the quasiparticle mass, which appears

in $\vec{v}_{\vec{k}\sigma}$. How did the quasiparticle mass get converted back to the bare mass? The answer is that the Landau interaction function, acting on the self consistent solution of Eq. (53) reinstates the bare mass according to the Galilean invariance relation discussed in section 2.5. Inclusion of the quasiparticle collision term does not change these results (if translational and Galilean invariance are in force), but affects the dispersion and introduces damping of the collective modes at finite wave vector.

The other classic application of the kinetic equation for quasiparticles is the calculation of the transport coefficients of a Fermi liquid. These are the spin diffusion constant, D_s, the shear and bulk viscosities, η and ζ respectively, and the thermal conductivity κ. D_s is the constant of proportionality between the spin current and the gradient of spin density that drives it. Similarly, η and ζ can be viewed as the traceless and traceful components of a diffusion tensor for the momentum density – with the momentum current being driven by a gradient in the macroscopic velocity field. Lastly, κ is the constant of proportionality between the heat (entropy) current and the gradient of temperature that drives it. Quasiparticle collisions, which are responsible for the finite quasiparticle lifetime, are absolutely essential to calculate these transport coefficient. In fact, these coefficients would all be infinite if those collisions were neglected, which of course becomes a better and better approximation as the temperature is reduced. This counterintuitive result (divergence of the transport coefficients for $T \to 0$) follows from the "asymptotic freedom" of the Landau quasiparticles in this limit. Indeed, one can show that the transport coefficient are qualitatively described by the following formulas

$$D_s \sim v_F^2 \tau_s \,, \quad \eta \sim S\tau_\eta \,, \quad \zeta \sim B\tau_\zeta \,, \quad \kappa = n c_v v_F^2 \tau_q \,, \tag{54}$$

where S and B, are, respectively, the high-frequency shear modulus and the bulk modulus (both on the order of $n\epsilon_F$) and c_v is the heat capacity (per particle) of the Fermi liquid. Here τ_s, τ_η etc... are transport relaxation times which are related to the quasiparticle lifetime (since they all depend on the same scattering probabilities) but are not identical with it or with each other. All these scattering times diverge in the limit of zero temperature as $1/T^2$ in three dimensions. In two dimensions the situation is more delicate as the scattering times associated with spin diffusion and thermal conductivity diverge as $1/(T^2 \ln T)$, while the scattering time associated with the viscosity continues to diverge as $1/T^2$ [14]. This is due to the fact that the scattering processes that are responsible for the logarithmic divergence have zero momentum transfer and therefore do not contribute to the transfer of momentum within the liquid. The divergence is eventually cut off when the quasiparticle mean free path becomes comparable to the macroscopic size of the system, at which point the coefficients lose their hydrodynamic significance. You might wonder why the density diffusion constant D_n does not appear in our list of transport coefficients. The answer is that in the absence of impurities or external potentials quasiparticle collisions cannot change the total particle current: this leads to an infinite conductivity and then, via the Einstein relation – which connects the conductivity to the diffusion constant – to an infinite diffusion constant! The physical interpretation of this surprising result is that a density imbalance in a Fermi liquid does not relax via a diffusion process, but through the emission of sound waves.

The transport coefficients (or, equivalently, the transport relaxation times defined by Eq. (54)) can be calculated with the help of the kinetic equation. For the neutral Fermi liquid ^{3}He this was first done by Abrikosov and Khalatnikov [13]. Explicit formulas for the transport coefficients in terms of angular averages of quasiparticle scattering probability are known for the three-dimensional case: see Eqs. (1.171) and (1.151) of Ref. [3]. The scattering probability is obtained from the Fermi golden rule, just as in the calculation of the quasiparticle lifetime. We refer the interested reader to Ref. [3] for details.

4 Microscopic basis of the Landau theory of Fermi liquids

Landau guessed the theory of interacting Fermi liquids largely on a basis of physical intuition. Shortly afterwards, it was shown that indeed the theory could be "derived" (or, more accurately, shown to be self-consistent) from the microscopic Hamiltonian under certain assumptions of continuity and regularity. Nowadays the Landau theory is recognized as an early example of *renormalization*, whereby the exact many-body Hamiltonian is transformed, through recursive elimination of fast degrees of freedom, into an effective Hamiltonian of weakly interacting quasiparticles. A "poor man" version of this theory, based on seminal work by Hamann and Overhauser [15], can be found in Section 8.6 of Ref. [2].
Without going into technical details, which can be easily found in the literature (including Ref. [2]), we summarize the main correspondences between the Landau theory and the microscopic theory.

4.1 Existence of quasiparticles and self-energy

The existence of long-lived quasiparticles, with an inverse lifetime that scales as $|k-k_F|^2$ for $k \to k_F$ corresponds to the fact that the microscopic retarded Green function has the form

$$G(\vec{k},\omega) \;=\; G^{(\mathrm{reg})}(\vec{k},\omega) + \frac{Z_{\vec{k}}}{\omega - \frac{\mathcal{E}_{\vec{k}}}{\hbar} + \frac{i}{2\tau_{\vec{k}}}}\,, \tag{55}$$

where $G^{(\mathrm{reg})}(\vec{k},\omega)$ is a regular function of $\vec{k}$ and ω, and we have omitted the spin dependence for simplicity. Thus, the retarded Green function is dominated by a single pole of strength $Z_{\vec{k}}$ ($0 < Z_{\vec{k}} < 1$) at the complex frequency $z = \frac{\mathcal{E}_{\vec{k}}}{\hbar} - \frac{i}{2\tau_{\vec{k}}}$ in the lower half of the complex plane (as required by causality). The imaginary part of the frequency at the pole implies an exponential decay, with a characteristic time $\tau_{\vec{k}}$, of the squared amplitude of a plane-wave state. The quasiparticle energy, the strength of the pole (also known as the *renormalization constant*), and the plane-wave lifetime are determined by the (retarded) self-energy function[11] as follows

$$\mathcal{E}_{\vec{k}} = \varepsilon_{\vec{k}} + \mathrm{Re}\,\Sigma^{\mathrm{ret}}_{\sigma}(\vec{k},\mathcal{E}_{\vec{k}})\,, \tag{57}$$

[11] We remind the reader that the self-energy is defined as the difference between the inverse noninteracting Green function and the inverse Green function:

$$\Sigma = G_0^{-1} - G^{-1} \tag{56}$$

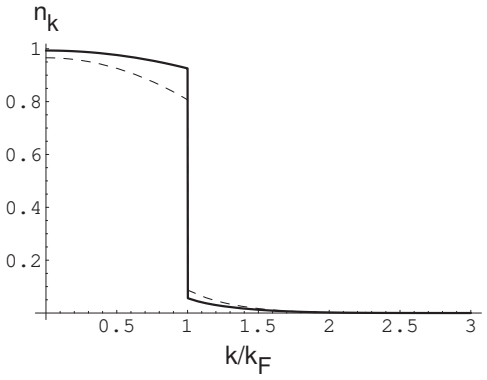

Fig. 10: *Behavior of the plane-wave states average occupation number in three dimensions at $r_s = 2$ (solid line) and $r_s = 5$ (dashed line). The jump of n_k at $k = k_F$ is the renormalization constant Z_{k_F}.*

where $\varepsilon_{\vec{k}}$ is the bare particle energy,

$$Z_{\vec{k}} \equiv \left(1 - \frac{1}{\hbar}\frac{\partial}{\partial\omega}\,\mathrm{Re}\,\Sigma^{\mathrm{ret}}(k,\omega)\Big|_{\hbar\omega=\mathcal{E}_{\vec{k}}}\right)^{-1}, \tag{58}$$

and

$$\frac{\hbar}{2\tau_{\vec{k}}} \;\equiv\; Z_{\vec{k}}\,\big|\,\mathrm{Im}\,\Sigma^{\mathrm{ret}}(k,\mathcal{E}_{\vec{k}\sigma})\big|\,. \tag{59}$$

Crucially, the validity of the Fermi liquid scenario requires that

$$\mathrm{Im}\,\Sigma^{\mathrm{ret}}(\vec{k},\mathcal{E}_{\vec{k}\sigma}) \sim -a\big(k-k_F\big)^2, \tag{60}$$

where a is a positive constant, and the approximate equality $\sim$ disregards the possibility of logarithmic terms $\ln|k-k_F|$. Microscopic calculations of the self-energy confirm that the asymptotic form (60) is, at the very least, self-consistent, i.e., the presence of a pole of the form (55) in the Green function guarantees the vanishing of the imaginary part of the self-energy, which in turn implies a divergence of the quasiparticle life time.[12]

The existence of an infinitely sharp quasiparticle peak of strength $Z_{\vec{k}\sigma}$ in the spectral function (defined as the negative of the imaginary part of the retarded Green function) implies that the momentum state occupation number $n_{\vec{k}}$ has a discontinuity as a function of k at $k = k_F$, the magnitude of the discontinuity being given by Z_{k_F}. This is shown in Fig. 10. The fact that the discontinuity occurs precisely at $k = k_F$ where k_F is related to density by the ideal gas relation, is known in the literature as *Luttinger's theorem*. Notice that $Z_{\vec{k}}$ and the ground state occupations $n_{\vec{k}\sigma}$ are "invisible" in the Landau theory of Fermi liquids, which abstracts from the detailed structure of the ground state. On the other hand, all the parameters of the Landau theory can be calculated from the microscopic theory following well-defined procedures. For

[12] It must be noted that the plane-wave lifetime of Eqs. (55) and (59) is not exactly the same as the lifetime of a quasiparticle and a quasihole. The inverse of the former differs from the inverse of latter by factors $1 - n_{\vec{k}}$ and $n_{\vec{k}}$ respectively. See Ref. [2] for details.

example, the effective mass of the quasiparticle works out to be

$$\frac{m^*}{m} = \frac{1}{Z_{k_F}\left(1 + \frac{m}{\hbar^2 k_F}\frac{\partial}{\partial k}\,\mathrm{Re}\,\Sigma^{\mathrm{ret}}(k,\mu)\Big|_{k=k_F}\right)}\,. \tag{61}$$

The results of several microscopic calculations of the effective mass of quasiparticles in the 2D and 3D electron gas are presented and critically discussed in Chapter 8 of Ref. [2].

4.2 Landau interaction function and scattering amplitude

What about the Landau interaction function? What is its representation in the microscopic theory? Considering Eq. (7) and the microscopic expression, Eq. (57), for the quasiparticle energy, combined with Eq. (58) for the renormalization constant we arrive at

$$f_{\vec{k},\vec{k}'} = Z_{\vec{k}} Z_{\vec{k}'}\,\frac{\delta\Sigma(\vec{k},\mathcal{E}_{\vec{k}})}{\delta\mathcal{N}_{\vec{k}'}}\,, \tag{62}$$

where we continue to ignore the spin for simplicity. The problem with this expression is that the quasiparticle occupation number $\mathcal{N}_{\vec{k}'}$ is not a well-defined microscopic quantity. However, one can calculate the change in the Green function that follows from a change in the corresponding occupation number $n_{\vec{k}'}$ of the non-interacting ground state from which the interacting state is supposed to arise under adiabatic switching-on of the interaction. According to Landau's hypothesis of continuity, such a change will result in an identical change of the quasiparticle occupation number, while at the same time propagating through the expression for the self-energy to produce the desired $\delta\Sigma$. The analysis is quite complex (see Section 8.5.5 of Ref. [2] for details) but the final result is simple, at least formally:

$$f_{\vec{k},\vec{k}'} = Z_{\vec{k}} Z_{\vec{k}'} \lim_{\omega\to 0}\lim_{q\to 0} \Gamma_{\vec{k}\mathcal{E}_{\vec{k}};\vec{k}'\mathcal{E}_{\vec{k}'}}(q,\omega)\,, \tag{63}$$

where $\vec{k}$ and $\vec{k}'$ lie on the Fermi surface and $\Gamma_{\vec{k}\mathcal{E}_{\vec{k}};\vec{k}'\mathcal{E}_{\vec{k}'}}(q,\omega)$ is the probability *amplitude* for the two particles with momenta $\vec{k}$ and $\vec{k}'$ and energies $\mathcal{E}_{\vec{k}}$ and $\mathcal{E}_{\vec{k}'}$ respectively to scatter against each other with momentum and energy transfers q and ω, where both q and ω tend to zero in the order specified by Eq. (63) (i.e., q must tend to zero before ω does). Notice that this is the scattering amplitude between two particles embedded in the many-body system. As such it has a very complex diagrammatic representation but it can, in principle be calculated by the methods of many-body theory. In the simplest approximation, i.e., first order perturbation theory, the calculation becomes trivial and we find

$$f_{\vec{k},\vec{k}'} = -v_{\vec{k}-\vec{k}'}\,, \tag{64}$$

where $v_{\vec{k}-\vec{k}'}$ is the Fourier transform of the Coulomb interaction.[13]

[13] This formula is valid for same-spin electrons. If the electrons have opposite spin the first order interaction is zero.

The Landau interaction function $f_{\vec{k},\vec{k}'}$, from which the Landau parameters are extracted, should not be confused with the effective interaction between quasiparticles which is used to calculate the quasiparticle lifetime and the collision term in the kinetic equation of Section 3.5. The essential difference, already evident from Eq. (63), is that the Landau interaction function is the limit of the microscopic scattering amplitude for zero momentum and energy transfer, with the additional specification that the momentum transfer q tends to zero before the frequency transfer ω. In contrast to this, the effective interaction determines the scattering amplitude for all momentum transfers such that $0 < q \simeq 2k_F$, which connect two wave vectors on the Fermi surface and, again, for small energy transfers $\omega \ll \epsilon_F$, such that the quasiparticles remain close to the Fermi surface. Even in the limit of small q this is different from the Landau interaction function because the energy transfer goes to zero first![14]

This leaves us with the problem of extending the Landau interaction function to finite q, thus generating what is known as an effective interaction between quasiparticles [16–19]. This is an extremely difficult task, even if one limits oneself to seeking a "local" interaction, which depends only on q and ω and is averaged over the quasiparticle momenta $\vec{k}$ and $\vec{k}'$ over the Fermi surface. A very useful concept in this context has been that of the dimensionless *many-body local field factors*, denoted by $G^{s(a)}(q)$. The local field factors were originally introduced to generate approximate expressions for the density and spin response functions of a Fermi liquid beyond the random phase approximation: this concept is described in detail in Section 5.4 of Ref. [2]. It is now understood [17] that the local field factors are in a very precise sense the finite-q extension of the $\ell = 0$ Landau parameters:

$$-\lim_{q\to 0} v(q)\, G^{s(a)}(q) = f_0^{s(a)}\,, \tag{66}$$

where $v(q)$ is the Fourier transform of the bare interaction (Coulomb interaction for an electron liquid).[15] Armed with this understanding and with a decent microscopic calculation of the local field factors (see Ref. [2], Appendix 11) it is relatively easy to construct a local effective interaction appropriate to the problem at hand.[16] The most famous effective interaction between quasiparticles is perhaps the Kukkonen-Overhauser (KO) interaction [16], which is especially suitable for calculations of the superconducting transition temperature. The explicit form of this interaction is

$$V^s_{\text{eff}}(q,\omega) = v(q) + \big(v(q)(1-G^s(q))\big)^2 \frac{\chi_0(q,\omega)}{1 - v(q)(1-G^s(q))\chi_0(q,\omega)}\,, \tag{67}$$

[14] It can be shown that the relation between the amplitudes calculated for $q \to 0$ with these two different orders of limits is

$$A_\ell^{s(a)} = f_\ell^{s(a)}/(1+F_\ell^{s(a)}) \tag{65}$$

where $F_\ell^{s(a)}$ are the Landau parameters and $A_\ell^{s(a)}$ is the ℓ-th component in the Legendre-polynomial expansion of the $q \to 0$ limit of the scattering amplitude in the symmetric (s) or antisymmetric (a) spin channel.

[15] The quantity $-v(q)\, G^{s(a)}(q)$, often generalized to finite frequency, is known in the density functional theory literature as the "exchange-correlation kernel".

[16] Different effective interactions arise depending on whether one consider the interaction between two particles extraneous to the Fermi liquid, e.g., two charged impurities in the electron gas, an extraneous particle and a quasiparticle, or two quasiparticles. Even in the latter case differences arise depending on if one wants to calculate a superconducting transition temperature or a self-energy, since different classes of diagrams are involved in the two cases.

for the spin-symmetric channel (s), and

$$V^a_{\rm eff}(q,\omega) = \big(v(q)G^a(q)\big)^2 \frac{\chi_0(q,\omega)}{1+v(q)G^a(q)\chi_0(q,\omega)}\,, \tag{68}$$

for the spin antisymmetric channel (a), where $\chi_0(q,\omega)$ is the well-known Lindhard function, i.e., the density/spin response function of the non-interacting electron gas, and $V^{s(a)}_{\rm eff} \equiv V^{\uparrow\uparrow}_{\rm eff} + (-)V^{\uparrow\downarrow}_{\rm eff}$. An application of the KO interaction to a study of superconductivity in an electron-hole liquid can be found in Ref. [18]. For a very recent application of this interaction to the study of superconductivity in elemental metals see Ref. [19].

5 Fermi liquid of massless Dirac fermions

We now discuss some peculiarities of the Fermi liquid formed by "massless Dirac fermions" (MDF). MDFs occur near the crossing of two bands, such as occurs in a single layer of C atoms (graphene) and in numerous Dirac and Weyl semimetals: the band dispersion is linear in k near the crossing point, which we set at $k = 0$ (see Fig. 11): $\epsilon_k = \hbar v k$ where v is the fermion velocity, independent of momentum. A Fermi liquid is realized when the Fermi level crosses one of the bands (say the upper one) in the vicinity of the crossing. The existence of a non-vanishing Fermi wave vector k_F, related to the density of electrons in the upper band by the usual Fermi liquid relation, $k_F^2 = 2\pi n$, and a non-vanishing Fermi energy $\epsilon_F = \hbar v k_F$, are sufficient to establish the existence of a normal Fermi liquid at sufficiently low temperature $k_B T \ll \epsilon_F$. This is because the structure of the low-energy excitations in the MDF system is indistinguishable from that of the low-energy excitations in an ordinary parabolic band:

$$\epsilon_k = \epsilon_F + \hbar v\big(k-k_F\big) = \epsilon_F + \hbar\frac{k_F}{m_c}\big(k-k_F\big)\,, \tag{69}$$

where $m_c \equiv \hbar k_F/v$, also known as the "cyclotron mass", plays the same role as the bare electron mass for parabolic bands. Unlike the bare electron mass, however, m_c is density-dependent, scaling as $n^{1/2}$ in 2D, and thus vanishing as the Fermi level approaches the crossing point for $k_F \to 0$. In view of this correspondence, we expect the Fermi liquid concept to be robust in this new situation, in particular we expect to find long-lived quasiparticles with an effective mass m^* that is somewhat different from m_c due to the presence of electron-electron interaction. Nevertheless there are some important differences to be kept in mind. We list them below.

1. First of all, the range of validity of the Fermi liquid theory shrinks to zero as the Fermi energy approaches the crossing point of the bands, also known as the "Dirac point". When $\epsilon_F = 0$ there is no Fermi surface and no Fermi liquid. The length and time scales provided by k_F^{-1} and $\hbar\epsilon_F^{-1}$ diverge, and the system becomes scale-invariant (or quantum-critical). The only energy scale left is $k_B T$ itself and the inverse lifetime of electrons and holes must be proportional to $k_B T$, which is of the same order of magnitude as the energy of the excitations. This defines the so-called "Planckian regime", and we see that the whole

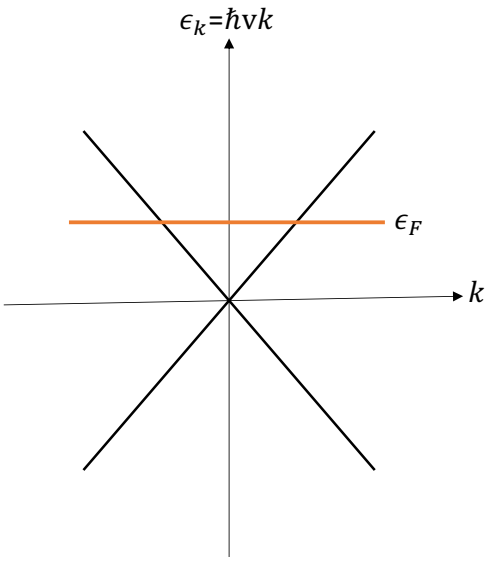

Fig. 11: *Band dispersion for massless Dirac Fermions in the vicinity of the crossing point (Dirac point). Also shown is the Fermi level which defines our Fermi liquid.*

concept of quasiparticle falls apart in this regime. The nature of the quantum critical state at $\epsilon_F = 0$ (also known as the charge neutrality point) is not completely understood at present. Strong electron-electron interactions have been dealt with by a hydrodynamic description, where individual particles are superseded by collective variables such as density and current. Alternatively, it has been suggested that electrons and holes in the upper and lower band bind together to produce a gapped state known as "excitonic insulator".

2. Although, for finite k_F, the structure of the low-energy excitations remains the same as in the standard parabolic case, there are some important differences in the structure of higher energy excitations. In particular, the presence of electrons in the fully occupied lower band cannot be ignored, as it gives a significant contribution to the Fermi liquid properties. These contributions fall into two categories: (i) contributions to the Landau interaction function arising from interactions between the electrons near the Fermi surface and those in the fully occupied bands, (ii) contributions to the quasiparticle lifetime arising from interband transitions at energies lower than or comparable to the Fermi energy.

   Concerning (i) it must be noted that the relative strength of the electron-electron interaction, as measured by the ratio of the potential energy to the kinetic energy, is no longer density-dependent: rather it becomes a fixed constant $\alpha = \frac{e^2}{\hbar v}$ of order 1. However, there is now another measure of the importance of interaction effects, and that is Λ/k_F where Λ is an ultraviolet cutoff wave vector, which determines the largest momentum of the occupied states for which the linear (massless) band model is still valid. This cutoff is poorly defined, but is expected to be of order $1/a$, where a is the lattice constant. So even though α is constant, interaction effects become stronger as k_F tends to zero, which is similar to the familiar situation, but leads to very different phenomenology in this case. For example it can be shown that the inverse compressibility, proportional to $\partial\mu/\partial n$,

is increased by interactions rather than decreased [20]. This happens because when the electronic density is increased the Fermi level in the upper band moves farther away from the lower band: the negative exchange energy that is lost due to this effect outweighs the negative exchange energy that is gained by having more electrons in the upper band. The same phenomenon is observed for the spin susceptibility, which is now suppressed, rather than enhanced, by interactions.

Concerning point (ii), a detailed analysis presented in Ref. [21] shows that the quasiparticle lifetime (in the upper band, $+$) is given by

$$\frac{1}{\tau^{(\mathrm{e})}_{\boldsymbol{k},+}} \simeq \frac{\varepsilon_{\mathrm{F}}}{\hbar}\frac{1}{\pi N(0)}\left(\frac{\xi_{\boldsymbol{k},+}}{\varepsilon_{\mathrm{F}}}\right)^2 \ln\left(\frac{\Lambda}{\xi_{\boldsymbol{k},+}}\right), \tag{70}$$

at zero temperature and

$$\frac{1}{\tau^{(\mathrm{e})}_{\boldsymbol{k},+}} \simeq \frac{\varepsilon_{\mathrm{F}}}{\hbar}\frac{\pi}{2N(0)}\left(\frac{k_{\mathrm{B}}T}{\varepsilon_{\mathrm{F}}}\right)^2 \ln\left(\frac{\Lambda}{k_{\mathrm{B}}T}\right), \tag{71}$$

at finite temperature. This is essentially the Giuliani-Quinn result [11] discussed earlier in Section 3.3. Three main differences with respect to the classic calculation for an ordinary two-dimensional electron gas have been identified in Ref. [21]: i) a simple Fermi golden rule approach with statically screened Coulomb interactions is not viable in graphene as it yields logarithmically-divergent intra-band scattering rates due to the collinear scattering singularity; ii) the leading-order contribution to the quasiparticle decay rate in the low-energy and low-temperature limits is completely controlled by scattering events with small momentum transfer: the $2k_{\mathrm{F}}$ contributions are suppressed by the chiral nature of massless Dirac carriers in graphene; iii) because of point ii), the leading order contribution to the quasiparticle decay rate is completely independent of the strength of the background dielectric constant ϵ: the result is therefore *universal* in that it does not depend on the substrate on which graphene is placed.

3. As the Fermi level approaches the crossing point with decreasing density, the Fermi liquid concept remains in force as long as $k_F > 0$, but the effective mass of quasiparticles is found to be logarithmically suppressed relative to the noninteracting cyclotron mass [22]. Another way of saying this is that the Fermi velocity v is renormalized to $v^*(k_F) > v$ where $v^*(k_F)$ diverges logarithmically in first order perturbation theory as $k_F \to 0$:

$$\frac{v^*(k_F)}{v} = 1 + \frac{\alpha}{4}\ln\left(\frac{\Lambda}{k_F}\right). \tag{72}$$

While an increase of the velocity, leading to a reshaping of the bands near the Dirac point, has been experimentally observed, the divergence of $v^*(k_F)$ poses a problem of legitimacy for the microscopic perturbation theory on which this prediction is based. Clearly a non-perturbative approach is needed to analyze the $k_F \to 0$ limit. This approach is provided by the renormalization group, which is reviewed in Ref. [22], and generally confirms the predictions of the weak coupling theory.

6 Non-Fermi-liquid behavior

6.1 Disordered electron liquid

Thus far we have only considered translationally invariant electron liquids. But real electron liquids are inevitably subjected to potentials that break translational invariance, such as the periodic potential in a crystal lattice, or the potential from randomly distributed impurities.

A general approach to inhomogeneous interacting Fermi liquids begins with a consideration of the exact eigenstates $|\phi_\alpha\rangle$ of a single particle – an electron in our case – in the external potential. These are Bloch waves in a perfectly periodic crystal lattice, but have no definite symmetry in the presence of a random impurity potential. In either case, the ground-state of the noninteracting system is obtained by singly occupying the eigenstates with the N lowest energies ε_α. The highest occupied eigenvalue defines a Fermi energy $\epsilon_F = \varepsilon_N$, but not a Fermi surface. Excited states are obtained by promoting some electrons from below the Fermi level to above the Fermi level. All these states are described by a set of occupation numbers $\mathcal{N}_\alpha$.

We now start from one of these states, and slowly turn on the electron-electron interaction, expecting to generate long-lived interacting states characterized by a quasiparticle distribution $\mathcal{N}_\alpha$. But here comes an important difference. While in the homogeneous case the interaction between quasiparticles is hindered by the Pauli exclusion principle *and* by the conservation of momentum and energy, in an inhomogeneous system the constraint of momentum conservation does not exist. It is therefore expected that the quasiparticles will interact more strongly and decay more rapidly than in the homogeneous case. How much more rapidly? This is an essential question, since the very existence of the Fermi liquid requires that the decay rate of a quasiparticle of energy ε tend to zero more rapidly than $\varepsilon - \epsilon_F$ when $\varepsilon \to \epsilon_F$.

The answer depends on whether the system is periodic or disordered. In periodic systems crystal momentum (the Bloch wave vector) is conserved up to reciprocal lattice vectors. While the occurrence of "umklapp" processes can alter the numerical value of the lifetime it does not lead to qualitative departures from the homogeneous picture.

In disordered systems the lack of momentum conservation has more serious consequences. First of all, if disorder is sufficiently strong, it can lead to *localization* of the quasiparticle states and hence change the electrical properties of the system from metal to insulator. We will not pursue this scenario here. Even in the weak disorder regime, i.e., for $k_F \ell \gg 1$, where ℓ is the electron mean free path, the combined effects of disorder and interactions can be significant. Because density fluctuations relax at a slower pace than in a perfect crystal the electrons within them stay together for a longer time and hence interact more strongly: this "electron loitering" leads to an enhanced quasiparticle decay rate. In three dimensions the decay rate goes as $\frac{1}{\tau^{(e)}} \sim (\varepsilon - \epsilon_F)^{3/2}$ and $\frac{1}{\tau^{(e)}} \sim (k_B T)^{3/2}$ in the limits of $k_B T \ll \varepsilon - \epsilon_F$ and $k_B T \gg \varepsilon - \epsilon_F$ respectively [23]. This is considerably larger than $(\varepsilon - \epsilon_F)^2$ and $(k_B T)^2$ yet still small compared to $\varepsilon - \epsilon_F$. In two dimension, on the other hand, one finds $\frac{1}{\tau^{(e)}} \sim (\varepsilon - \epsilon_F)$ at $T = 0$, and $\frac{1}{\tau^{(e)}} \sim (k_B T) \ln(k_B T)$ at finite temperature [24, 23], which implies that the conventional Fermi liquid picture is, at best, marginally valid.

In concomitance with the enhancement of the quasiparticle decay rate, the single-particle density of states, $N(\varepsilon)$, is *reduced* in the vicinity of the Fermi level, going as $(\varepsilon-\epsilon_F)^{1/2}$ in three dimensions, and $|\ln(\varepsilon-\epsilon_F)|$ in two dimensions [25]. These results are based on perturbation theory, and therefore cannot be trusted when the correction to the unperturbed density of states becomes too large (e.g., in two dimensions, for $\varepsilon \to \epsilon_F$). Notice, however, that a suppression of the density of states at the Fermi level is also predicted in the limit of strong disorder, even though the physics appears to be quite different in that regime. Other thermodynamic and transport properties, such as the heat capacity and the conductivity, show non-analytic behavior in T at low temperature. Remarkably, there is no non-analytic correction to the compressibility.[17]
Going beyond perturbation theory, in the true spirit of the Landau theory of Fermi liquid, has turned out to be an extremely difficult task, and the problem remains unsolved to date (for a relatively recent review, see Ref. [27]). Suffice it to say that the scattering amplitudes in the antisymmetric spin channel have been predicted to diverge in the low energy sector, suggesting a diverging spin susceptibility associated with the formation of ferromagnetic domains at the length scale of the divergence. At the same time the inverse conductance is seen to initially increase, as if the system were going towards an insulating state, but it then reaches a maximum and begins to decrease as the divergence of γ^a takes over. These findings show very clearly that the noninteracting theory of localization needs serious revision in the presence of electron-electron interactions.

6.2 Luttinger liquid

An interesting phenomenon, known by the colorful name of "orthogonality catastrophe", lies at the heart of the distinction between Fermi liquids and non Fermi liquids. Imagine injecting an extra electron into the ground-state of a strongly correlated N-electron system. Since the new electron lacks the appropriate correlations with the pre-existing electrons, the state of the $N+1$-electron system after the injection is essentially orthogonal to the ground-state. In mathematical terms, the orthogonality catastrophe implies that the renormalization constant Z, defined in Section 4, vanishes in the thermodynamic limit.[18] We take this to be the defining feature of a non-Fermi liquid state.
A classic example of non-Fermi liquid behavior is the so-called "Luttinger liquid" [28], which is realized in quasi-one dimensional electronic systems such as Bechgaard salts, TTF-TCNQ, and carbon nanotubes, as well as in confined systems of fermionic cold atoms. The reduced effective dimensionality of these systems hinders single particle motion to the point that the particles must be regarded as strongly correlated even when their interactions are weak.
An immediate consequence of this situation is the disappearance of the quasiparticle δ-function peak in the spectral function $A(k_F, \omega)$ at the chemical potential: there are no single-electron

[17] It should mentioned, for completeness, that additional non-analytic corrections to the density of states and the specific heat arise when the transverse electromagnetic interaction between the electrons is taken into account. See Ref. [26] for details.

[18] It can be shown that the renormalization constant is the square of the overlap between the excited state of the system immediately after the injection of an electron and the ground-state of the system.

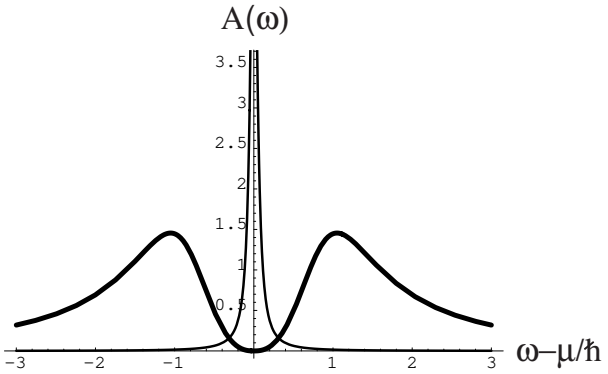

Fig. 12: *Schematic behavior of the local spectral function at $A(k_F, \omega)$ for a Luttinger liquid in the weak coupling regime (thin line) and in the strong coupling regime (thick line). In both cases $A(k_F, \omega) \sim \frac{1}{|\omega - \mu/\hbar|^\nu}$ for $\omega \to \mu/\hbar$, with the exponent ν tending to 1 for weak interactions and becoming negative for strong interactions. Notice the absence of the quasiparticle δ-function peak at $\omega = \mu/\hbar$.*

quasiparticles. For weak interactions the δ-function peak is replaced by a power-law divergence for $\omega \to \mu/\hbar$. With increasing coupling strength a sort of energy gap develops, whereby $A(k_F, \omega)$ vanishes with a power law for $\omega \to \mu/\hbar$ as shown in Fig. 12. The position of the lateral maxima in the spectral function is a rough measure of the energy of the disturbance created by the injection of the new electron in the liquid, while the "width" of these maxima is inversely proportional to the time needed for the many-electron system to adjust to the presence of the new electron. Another consequence of the vanishing of Z is that the plane-wave occupation number n_k is no longer discontinuous at $k = k_F$, even though a singularity persists in its derivatives with respect to k.

Luttinger liquids exhibit anomalous transport properties. For example, the electrical conductivity is expected to vanish at zero temperature. One might find this not so surprising since it is known that in a one dimensional system any amount of random disorder causes localization of the one-electron states, and hence a vanishing conductivity at $T = 0$. But, in the Luttinger liquid any perturbation that breaks translational invariance, e.g., *even a single impurity*, leads to an insulating state at $T = 0$. The physical reason for this effect is that the perfectly clean system is on the verge of spontaneously forming a charge density wave (CDW) of wave vector $2k_F$. Under these conditions even a single impurity can pin down an insulating CDW state.

6.3 Fractional quantum Hall liquid

It is still an open question whether non-Fermi liquid behavior occurs in more than one dimension. Although many theories suggest that this should be the case, for example in the normal phase of the superconducting cuprates, the experimental evidence remains somewhat ambiguous. A notable exception is the fractional quantum Hall liquid which occurs in the two-dimensional electron liquid at high magnetic field. This system presents the most radical departures from Landau Fermi liquid theory so far encountered in any condensed matter system, and the experimental evidence is extremely strong.

The basic reason for this exotic behavior is the *quenching of the kinetic energy* at high magnetic field. The kinetic energy of an electron in a magnetic field B is quantized in units of $\hbar\omega_c$ (where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency) so that its admissible values are

$$\epsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_c \tag{73}$$

where n is a non-negative integer. These energy levels are known as *Landau levels*. The number of degenerate states in each Landau level is proportional to the magnetic field and to the area ($\mathcal{A}$) of the system, and is given by

$$N_L = \frac{eB\mathcal{A}}{hc}\ . \tag{74}$$

Thus, in the limit of large magnetic field all the electrons fall in the lowest Landau level and the kinetic energy becomes a constant $N\frac{\hbar\omega_c}{2}$, where N is the number of electrons. The key parameter that controls the properties of the system is the *filling factor*

$$\nu = \frac{N}{N_L}\ , \tag{75}$$

which is less than 1 for a fractionally filled lowest Landau level. In this situation, the non-interacting ground-state is highly degenerate because the electrons can be distributed in many different ways among orbitals of the lowest Landau level. The degeneracy is removed, however, by the interaction. Under these conditions, the concept of an adiabatic connection between non-interacting and interacting states, which is central to the Landau theory of Fermi liquids, has no meaning, since the non-interacting limit of an interacting state is a completely ill-defined notion. Indeed, the incompressible quantum Hall liquid state, introduced by Laughlin [29] to explain the quantum Hall effect at filling factors of the form $\nu = \frac{1}{2k+1}$, where k is a positive integer, is not connected in any obvious way to a non-interacting state. The many-body wave function that describes it has multiple zeroes (of order $2k+1$) on the hypersurfaces of configuration space on which two electrons come in contact: in this sense, the zeroes of the wave function are "bound" to the particles. These multiple zeroes are far more powerful than the simple zeroes required by the Pauli exclusion principle in a non-interacting, and are ultimately responsible for giving the Laughlin state a particularly low interaction energy.

An exotic connection to a non-interacting state continues to exist, however, as has been pointed out by Jain, based on the beautiful idea of *composite fermions* [30]. Composite fermions are constructed by attaching to each electron an infinitely thin flux tube carrying an even number $2k$ of magnetic flux quanta $\Phi_0 \equiv \frac{hc}{e}$. These composite particles, like the original electrons, are subject to the external magnetic field B and interact with each other via the Coulomb interaction. The vector potential produced by the magnetic flux tubes exerts no force on the particles, and therefore does not contribute to the particle-particle interaction. When two composite particles are adiabatically interchanged along a path that does not enclose other particles the wave function is multiplied by an Aharonov-Bohm phase factor $e^{i\pi(2k+1)} = -1$: this indicates that the composite particles are fermions and the problem of interacting composite particles is mathematically identical to the original problem of interacting electrons. While the transformed

problem appears at first sight more difficult than the original, the advantage of this description appears when one introduces a mean field approximation. To this end, the fictitious magnetic fluxes attached to the particles are spread out, creating a uniform and constant magnetic flux proportional to the particle density. This average magnetic flux combines with the external physical flux, yielding an effective flux

$$B^*\mathcal{A} = B\mathcal{A} - 2k\Phi_0 N \tag{76}$$

where the flux tubes have been assumed to be antiparallel to the external field for $k > 0$ and parallel for $k < 0$. The composite fermions feel the effective field B^* and their effective filling factor ν^* is therefore given by (see Eqs.(74), (75) and (76))

$$\frac{1}{\nu^*} = \frac{1}{\nu} - 2k \; . \tag{77}$$

Thus, by cleverly choosing the values of $2k$ and ν, it is possible to convert a strongly-correlated problem of electrons at fractional filling ν into a weakly correlated problem of composite fermions at *integral* filling factor ν^*. For example if $\nu = \frac{1}{3}$ and $2k = 2$, then we have $\nu^* = 1$, which corresponds to a full Landau level of composite fermions. But the non-interacting ground-state at integer filling factor is perfectly well defined (one simply has to completely fill an integer number of Landau level) and provides a suitable starting point for an analytic continuation when the interactions are turned on. Although this conversion is inspired by a questionable mean field approximation, it provides a conceptual framework for constructing correlated wave functions of excellent quality *for electrons* starting from uncorrelated wave functions for composite fermions. The composite fermion picture also establishes a beautiful connection between the seemingly different phenomena of the integral and fractional quantum Hall effects. According to this picture, the fractional quantum Hall effect of electrons is nothing but the integral quantum Hall effect of composite fermions.

The special case of a half-filled Landau level ($\nu = \frac{1}{2}$) provides an extreme example of the power of the composite fermion idea. In this case, Eq. (77), with $2k = 2$, yields $\nu^* = \infty$, implying that the composite fermions experience no magnetic field on the average (i.e., the number of occupied Landau levels of composite fermions is infinite). If the mean field approximation makes sense one would then expect to see here some of the characteristic signatures of a two-dimensional Fermi liquid, such as a two-dimensional Fermi surface [31]. In fact, surface acoustic wave propagation experiments have provided considerable evidence in support of this prediction. However, one must keep in mind that the "Fermi liquid" behavior is limited to density and current response properties: it certainly does not apply to single-particle properties. For example, tunneling experiments show a pseudogap in the spectral density of one-electron excitations, characteristic of an "orthogonality catastrophe", and no sign of the quasiparticle peak characteristic of an ordinary Fermi liquid.

The *edge* of a 2D electron liquid at high magnetic field provides yet another example of non Fermi liquid behavior. It can be shown that the collective oscillations of the density in such an edge are dynamically equivalent to the collective oscillations of a *chiral Luttinger Liquid* [32],

i.e., a Luttinger liquid in which, after diagonalizing the electron-electron interaction, only one half of the bosons – those propagating to the right *or* those propagating to the left – are retained. Unlike previous putative realizations of the Luttinger liquid paradigm, the edges of the 2D electron liquid are essentially free of disorder effects and, more importantly, the Luttinger liquid coupling constant coincides with the bulk filling factor ν. This has allowed a rather detailed experimental verification of the non-universal exponents in the power-law decay of the correlation functions of the Luttinger liquid. Studies of the tunneling current between edges in the fractional quantum Hall regime have also provided the first convincing evidence of fractionally charged excitations in condensed matter systems.

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