The Beresinskii-Kosterlitz-Thouless transition: the two-dimensional world and its peculiarities

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Theory

- @ Rome
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Experiments

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- Tom Lemberger (Ohio, USA)
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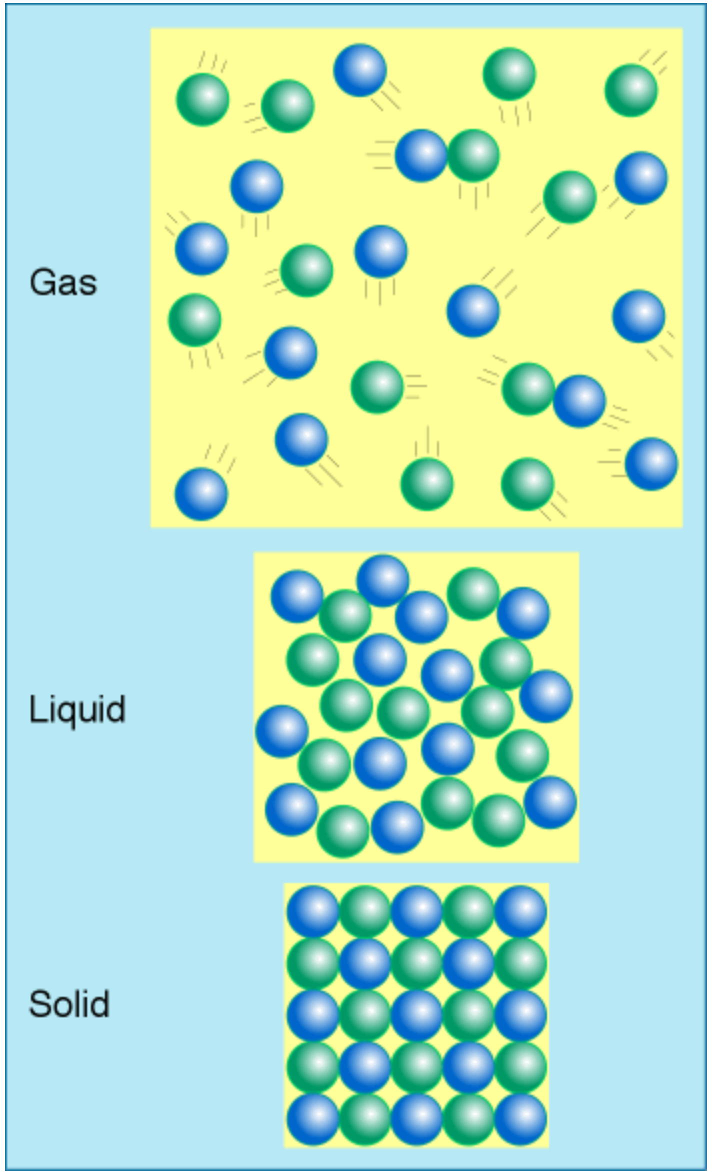
- Nicolas Bergeal (EPSI, Paris, France)

Phase transitions, order and rigidity

- Gas-liquid-solid: gas and liquid are not so different
- Solid IS different: it is highly ordered
  - “Broken translational symmetry”
- “Everyday life”: solid is rigid, liquid is not!

Phase transition:

new “order” and new “rigidity”

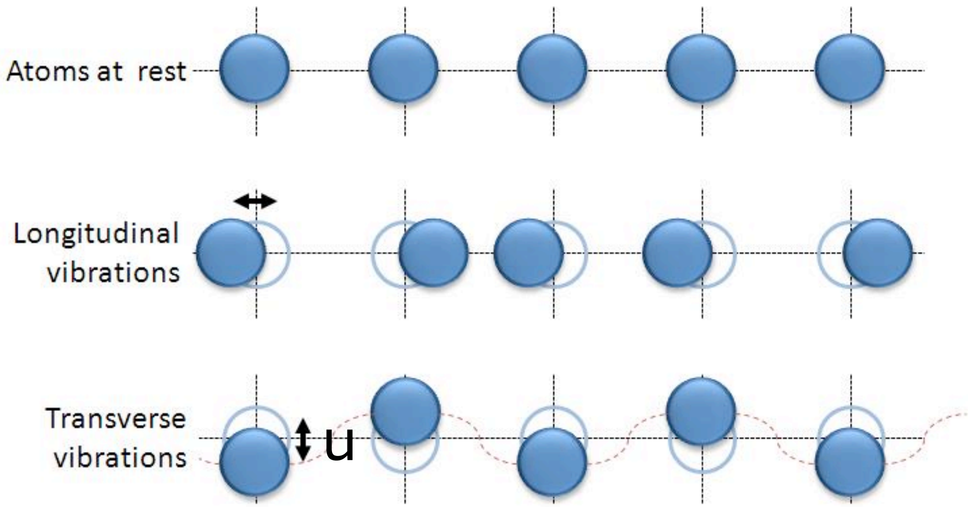


Liquid-solid transition

- Order parameter: average positions of the atoms at rest

- Rigidity: if one tries to move them away it costs energy. The elasticity modulus is the manifestation of the solid rigidity

- At long wavelength it costs nothing, since it is like displacing the crystal as a whole.



$$H = \frac{\kappa}{2}\int d\mathbf{r}(\nabla u)^2 = \frac{\kappa}{2}\int d\mathbf{q}\,\mathbf{q}^2 u_{\mathbf{q}}^2$$

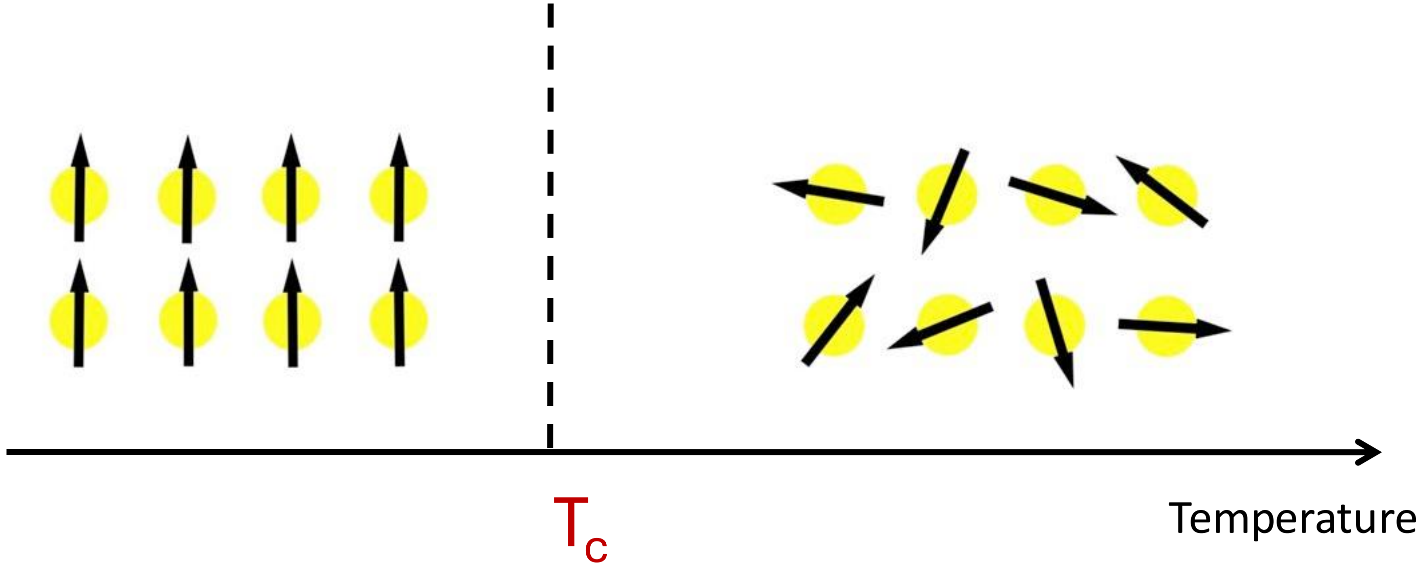
Acoustic phonons are the Goldstone modes emerging after breaking the ***continuous*** translational symmetry

Landau's view on phase transitions

- Interacting magnetic moments

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

- Having all the spins parallel minimizes the energy. So, at low enough temperature they pick up a preferential direction



"Spontaneously broken symmetry"

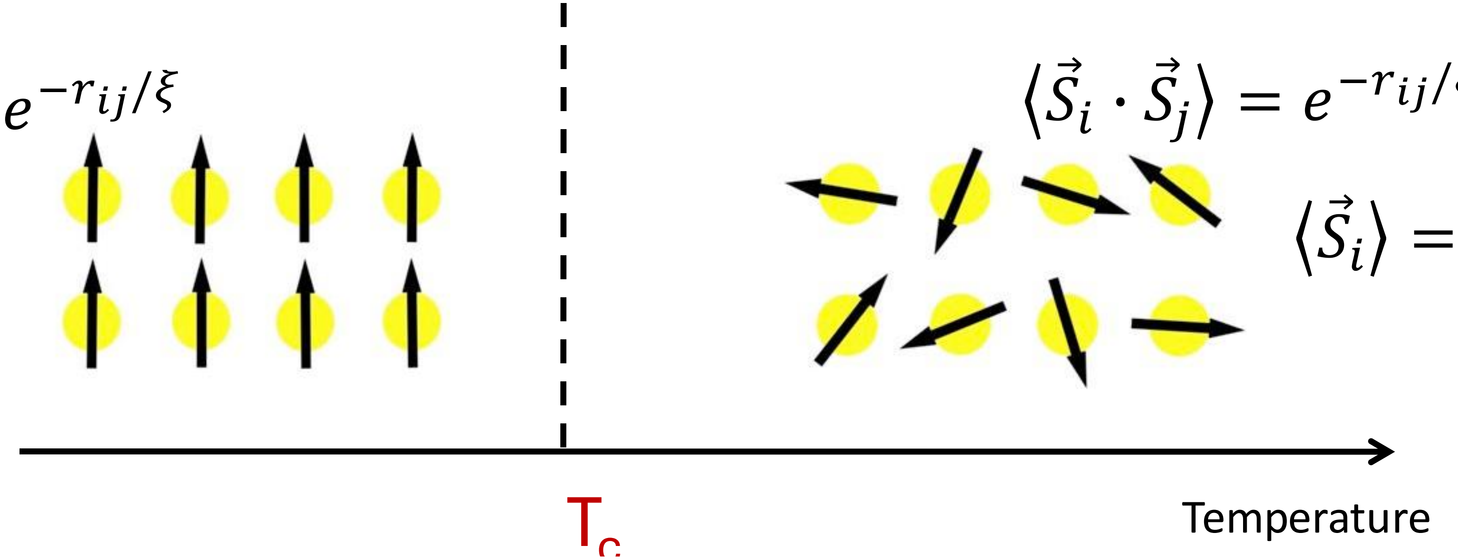
Landau's view on phase transitions

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle = \vec{M}_0^2 + e^{-r_{ij}/\xi}$$

$$\langle \vec{S}_i \rangle = \vec{M}_0$$

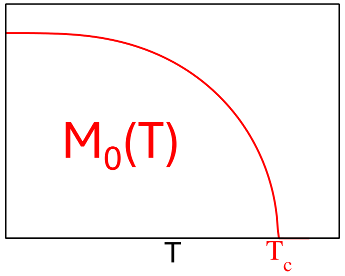
$$\langle \vec{S}_i \cdot \vec{S}_j \rangle = e^{-r_{ij}/\xi}$$

$$\langle \vec{S}_i \rangle = 0$$



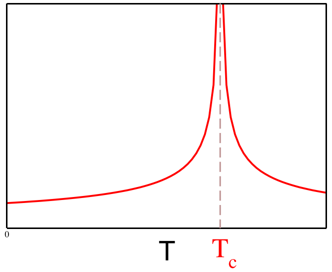
- Rotational symmetry is broken: the spins order in a preferential direction.

$$\langle \vec{S}_i \rangle = \vec{M}_0$$



- The correlation length diverges as one approaches T_c from both sides

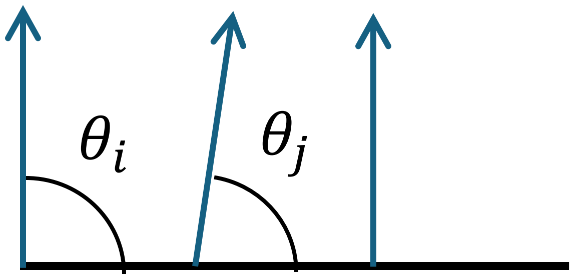
$$\xi_{\pm}(T) \sim \frac{1}{|T - T_c|^{\nu}}$$



what becomes "rigid"?

Order and rigidity: spin stiffness

- Classical spins confined in a plane

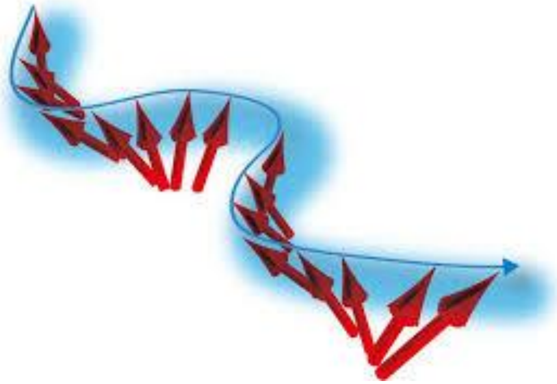


$$\mathbf{S}_i = S e^{i\theta_i} \qquad S = 1$$

$$H_{XY} = -J \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

- Let us consider a smooth change of the equilibrium configuration

$$H_{XY} \simeq \frac{J}{2} \int d^2 r \left[(\nabla\theta)^2 \right]$$



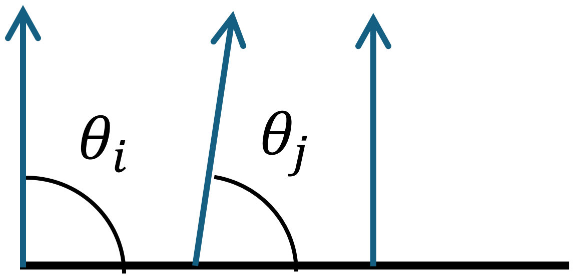
Rigidity: twist of spin at one end is transmitted to the other end (like "push" and "torsion" in a solid) J= spin stiffness

Like elastic energy: spin waves are the Goldstone mode emerging after breaking the continuous rotational symmetry

What is peculiar in *two dimensions*?

- Classical spins confined in a plane

$$\mathbf{S}_i = S e^{i\theta_i} \qquad S = 1$$



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- Let us consider a smooth change of the equilibrium configuration

$$\frac{H_{XY}}{T} \simeq \frac{J}{2T} \int d\mathbf{r} (\nabla\theta)^2$$

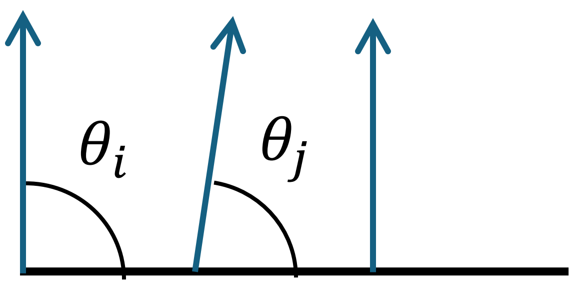
$$\langle \mathbf{S}_i \rangle = \langle e^{i\theta_i} \rangle = 0$$

Goldstone modes emerging after breaking a continuous symmetry destroy long-range order
Mermin-Wagner theorem (1966)

What is peculiar in *two dimensions*?

- Classical spins confined in a plane

$$\mathbf{S}_i = S e^{i\theta_i} \qquad S = 1$$



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$$\frac{H_{XY}}{T} \simeq \frac{J}{2T} \int d\mathbf{r} (\nabla\theta)^2 = \frac{J}{2T} \int \frac{d\mathbf{q}}{(2\pi)^2} \mathbf{q}^2 |\theta_\mathbf{q}|^2 \qquad \langle \theta_q^2 \rangle = \frac{T}{Jq^2}$$

$$\langle \mathbf{S}_i \rangle = \langle e^{i\theta_i} \rangle = e^{-\langle \theta_i^2 \rangle / 2} \qquad \langle \theta_i^2 \rangle = \int d^2q \, \langle \theta_q^2 \rangle \sim \frac{T}{J} \int_{\frac{1}{L}}^{\frac{1}{a}} dq \, \frac{1}{q} \sim \frac{T}{J} \ln \frac{L}{a}$$

All BKT physics has to do with log-like interactions in 2D

Correlation functions and phase transition

- Let us look at the correlation functions

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = \langle e^{i(\theta_i - \theta_j)} \rangle$$

Measures the relative orientation between distant spins

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \sim \left(\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \right)^{T/2\pi J}$$

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \sim e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}$$

Temperature

Low T, Gaussian approximation

$$H_{XY} \simeq \frac{J}{2} \int d^2 r \left[(\nabla \theta)^2 \right]$$

A “stiffness” manifests in the slow (power-law) decay of the correlation function

High T, full cosine form

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Distant spins are completely uncorrelated, the system is disordered

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T_{BKT} Temperature

Low T, Gaussian approx

The change of behavior must occur with a transition in between

ull cosine form

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Which kind of phase transition?

- Let us look at the correlation functions

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$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \sim e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}$$

T_{BKT}

- The purely Gaussian model is not enough to describe it
- The full cosine Hamiltonian admits one additional ***discrete*** symmetry

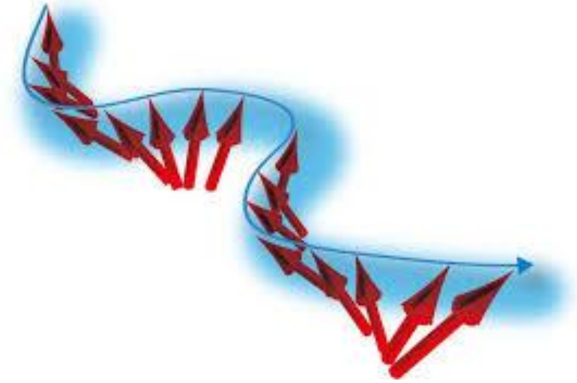
$$\theta_i \to \theta_i \pm 2\pi$$

- Vortex (topological) excitations are also present

$$\oint \nabla\theta = \pm 2\pi$$

Vortex and topology

- A spin waves can be “reduced” to the ordered state
- A vortex configuration cannot



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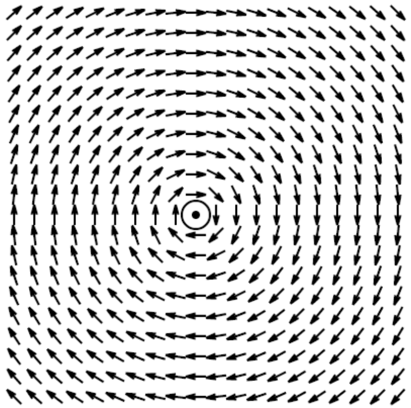


The energetic balance

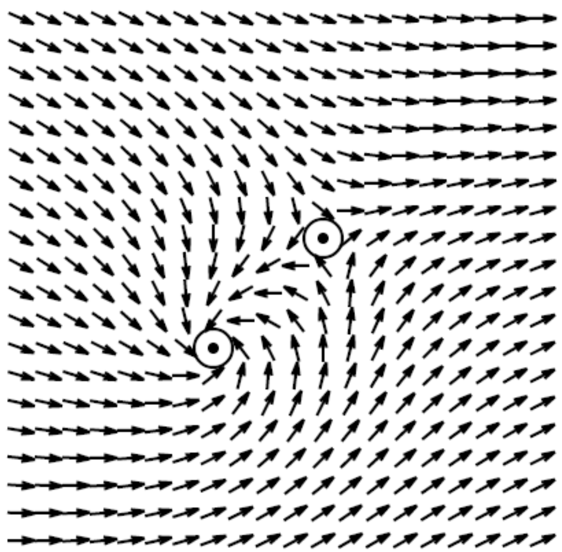
- Single-vortex energy balance

K&T J Phys.C 1972

$$F = E - TS :$$

$$E = \frac{J}{2}\int d\mathbf{r}(\nabla\theta_V(\mathbf{r}))^2 = \frac{J}{2}\int_a^L 2\pi dr\, r\frac{1}{r^2} = \pi J \log\frac{L}{a}$$

- On the other hand, bound pairs of vortices of opposite vorticity have a finite energy: a finite T one has a finite probability of their occurrence



The energetic balance

- Single-vertex energy balance

K&T J Phys.C 1972

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$$S = \log(\#\text{positions}) = \log\left(\frac{L}{a}\right)^2$$

$$F = E - TS = (\pi J - 2T)\ln\frac{L}{a}$$

At $T_{BKT} \approx \frac{\pi J}{2}$ the entropic gain makes free vortices possible
When vortices proliferate the system looses suddenly spin rigidity: phase transition

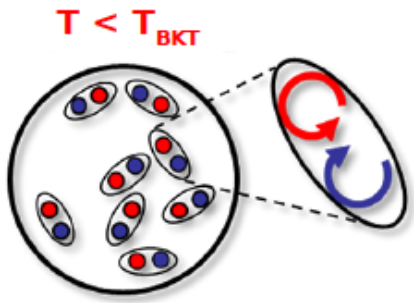
A more redifed treatment takes into account interactions between vortices

Coulomb-gas mapping

- Vortices → $q_i = \pm 1$ charges
- Interaction between vortices: $V(r) \sim J\, ln(r)$
- Only neutral configurations are allowed

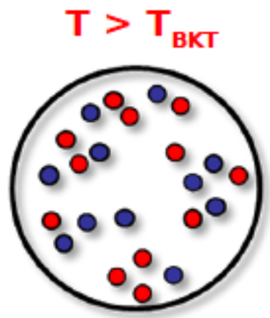
$$\sum_i q_i = 0 \qquad H_V = \mu \sum_i q_i^2 - \pi J \sum_{i \neq j} ln\left(\frac{r_{ij}}{a}\right) q_i q_j \qquad \mu = \frac{\pi^2 J}{2}$$

- Low T: bound charges, log potential = power-law decaying correlations
- High T: free charges, screened log potential = exponentially decaying correlations



Bound vortex-antivortex pairs

Vortex unbinding transition as insulator-metal transition in 2D



Proliferation of free vortices

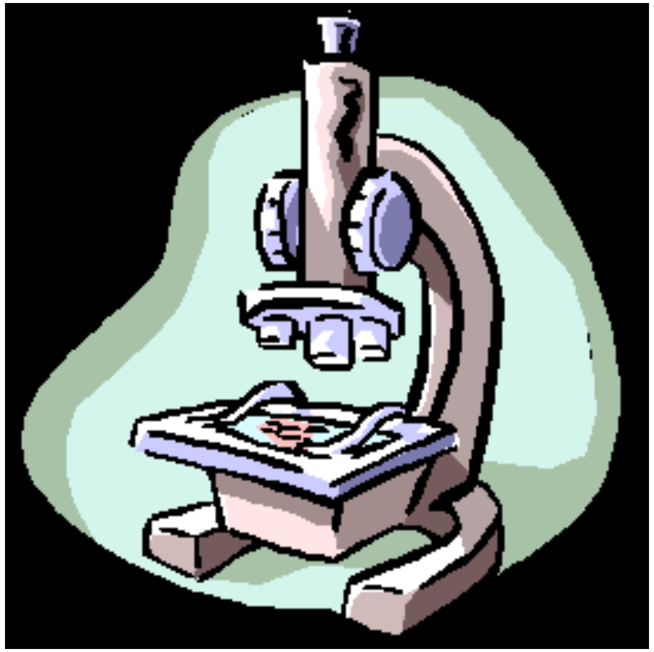
The BKT Renormalization Group equations

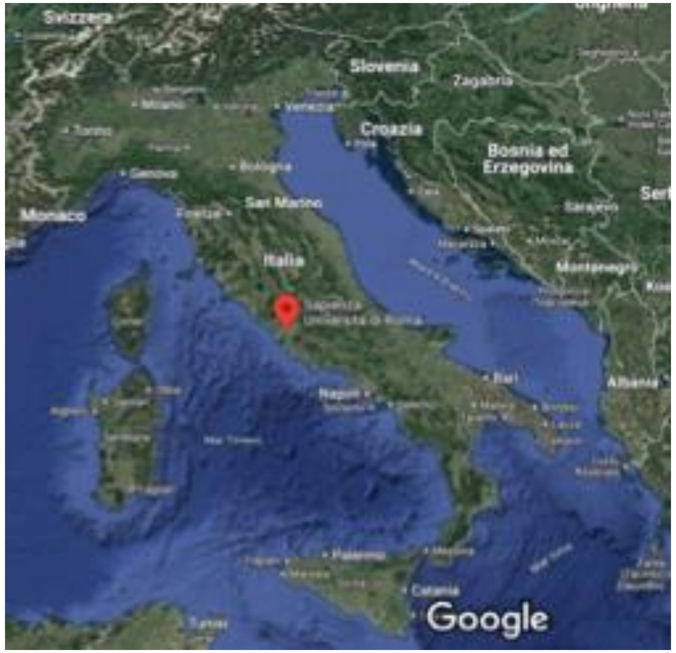
- All these arguments are made more precise by the RG approach

J.M.Kosterlitz, (1974)

We start from $H(a_0)$ and we aim at deriving $H(a \to \infty)$, like changing the magnifying power of a microscope

$$\ell = log(a/a_0)$$





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$$\ell = log(a/a_0)$$

$a = a_0$

$$H = \frac{J}{2}\int d^2r\,[(\nabla\theta)^2] + H_V$$

$a \to \infty$

low T

$$H_{eff} = \frac{J_s}{2}\int d^2r\,[(\nabla\theta)^2]$$

Finite rigidity

$$K = \frac{\pi J_s}{T}$$

$a \to \infty$

high T

$$H_{eff} = H_V$$

Zero rigidity

$$g = e^{-\beta\mu}$$

The BKT Renormalization Group equations

- Couplings: K and vortex fugacity y

$$K = \frac{\pi J}{T} \qquad g = e^{-\beta\mu} \qquad \mu_{XY} = \frac{\pi^2 J}{2}$$

$$\frac{dK}{d\ell} = -K^2 g^2,$$
$$\frac{dg}{d\ell} = (2-K)g.$$

- Superfluid stiffness:

$$J_s \equiv \frac{TK(\ell \to \infty)}{\pi}$$

Any additional effect entering $J_s(T)$ besides vortices should be added in $K(\ell = 0)$
In the XY model also *anharmonic spin waves* contribute to suppress linearly $J_s(T)$

$$K(\ell = 0) = \frac{\pi J}{T}\left(1 - \frac{T}{4J}\right)$$

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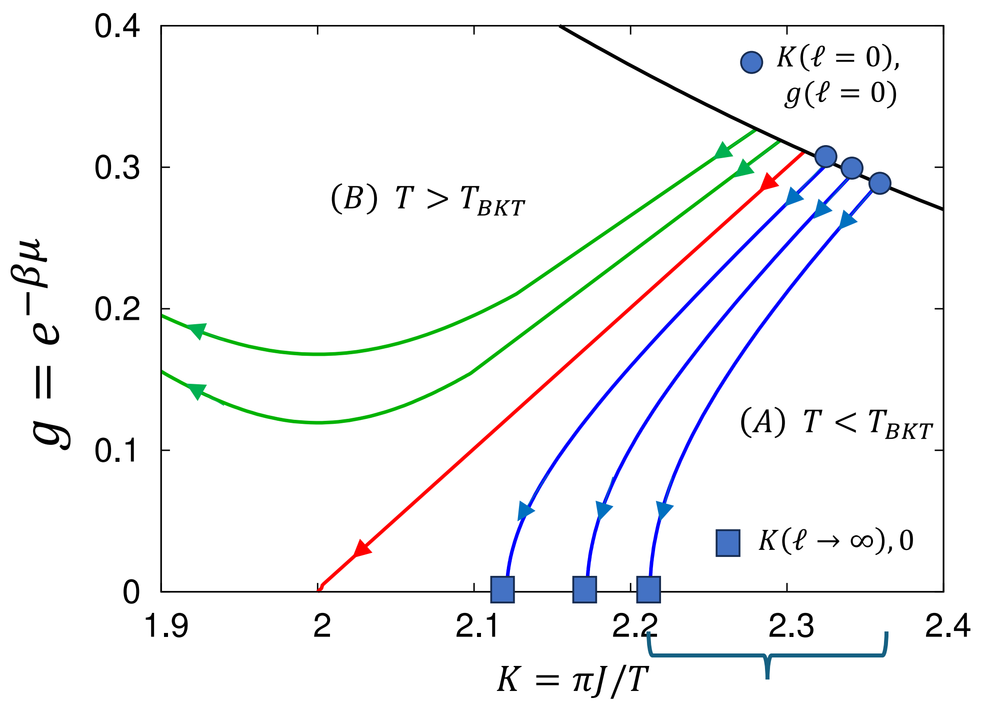

- Superfluid stiffness:

$$J_s \equiv \frac{T K(\ell \to \infty)}{\pi}$$

T< T_{BKT} : $J_s \approx J$

=2 at T_{KT}

T> T_{BKT} : J_s =0

The difference between $K(\ell = 0)$ and $K(\ell = \infty)$ depends on μ /J , that controls the density of bound vortex pairs at the smallest scale

The BKT Renormalization Group equations

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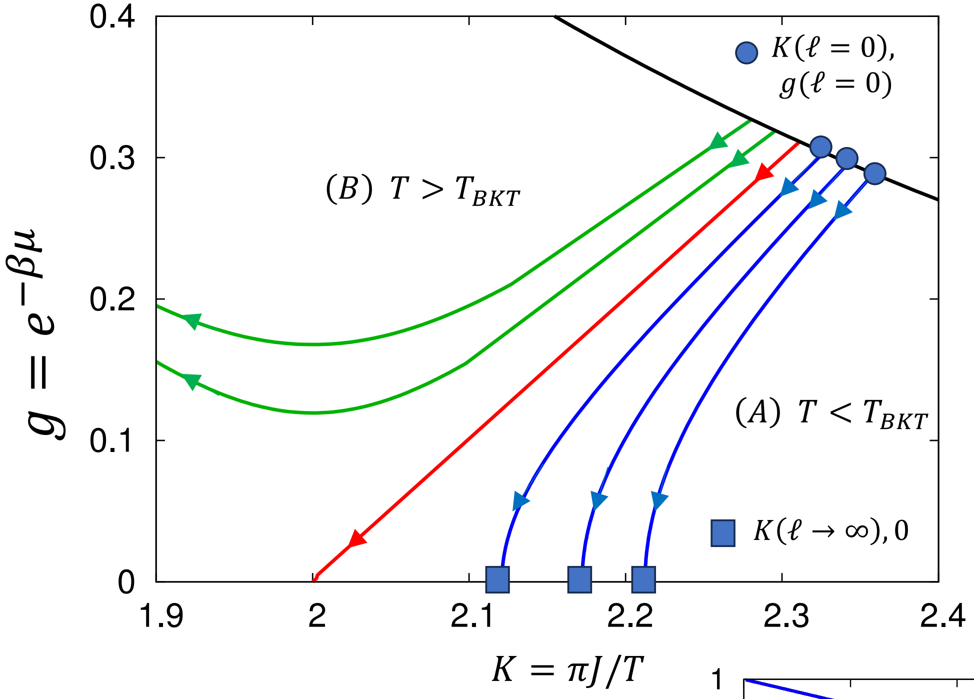

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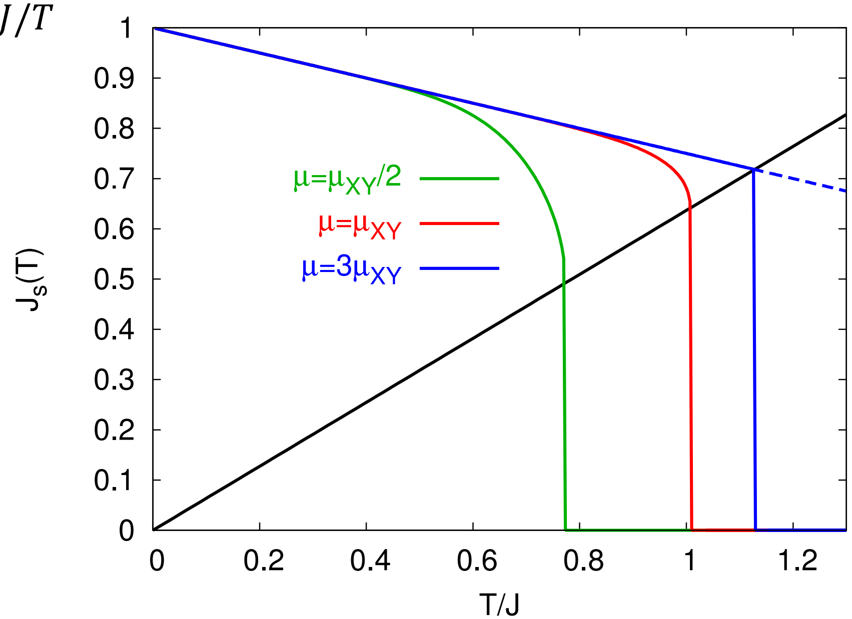

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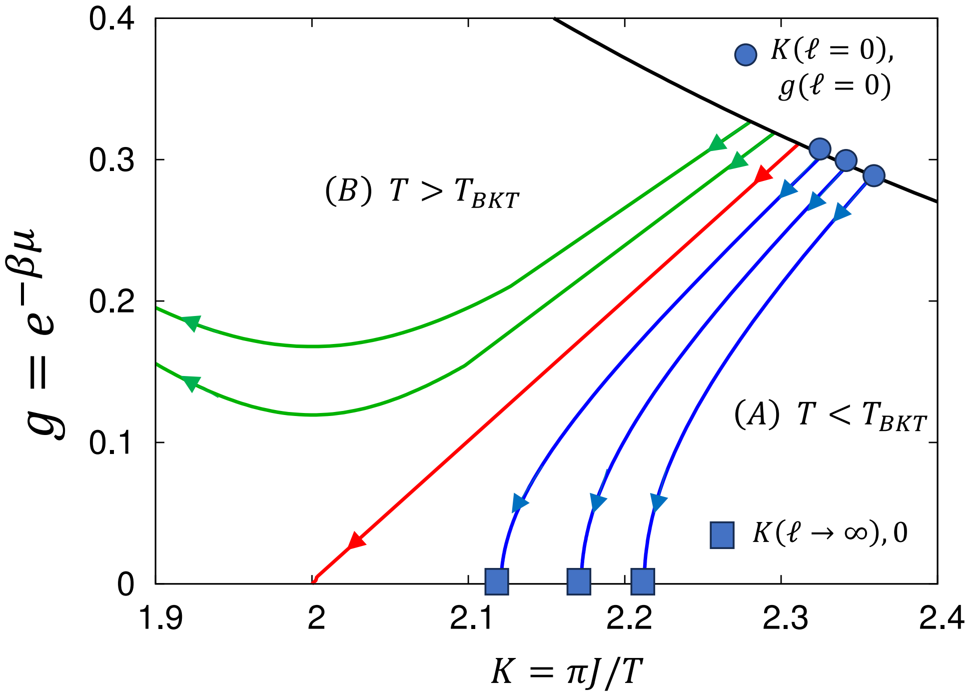


- Correlation length:

$$\boxed{\xi^{+}_{BKT} = a e^{b/\sqrt{t}}} \qquad t = \frac{T - T_{BKT}}{T_{BKT}} \qquad T < T_{BKT}$$

$$\boxed{\xi^{-}_{BKT} = \infty} \qquad T > T_{BKT}$$

Markedly different from ordinary phase transitions

$$\boxed{\xi^{\pm} \sim \frac{1}{|T - T_c|^{\nu}}}$$

So far, classical spins in two dimensions

However, the real success of BKT physics comes from its relevance for superfluid/superconducting systems in quasi two dimensions

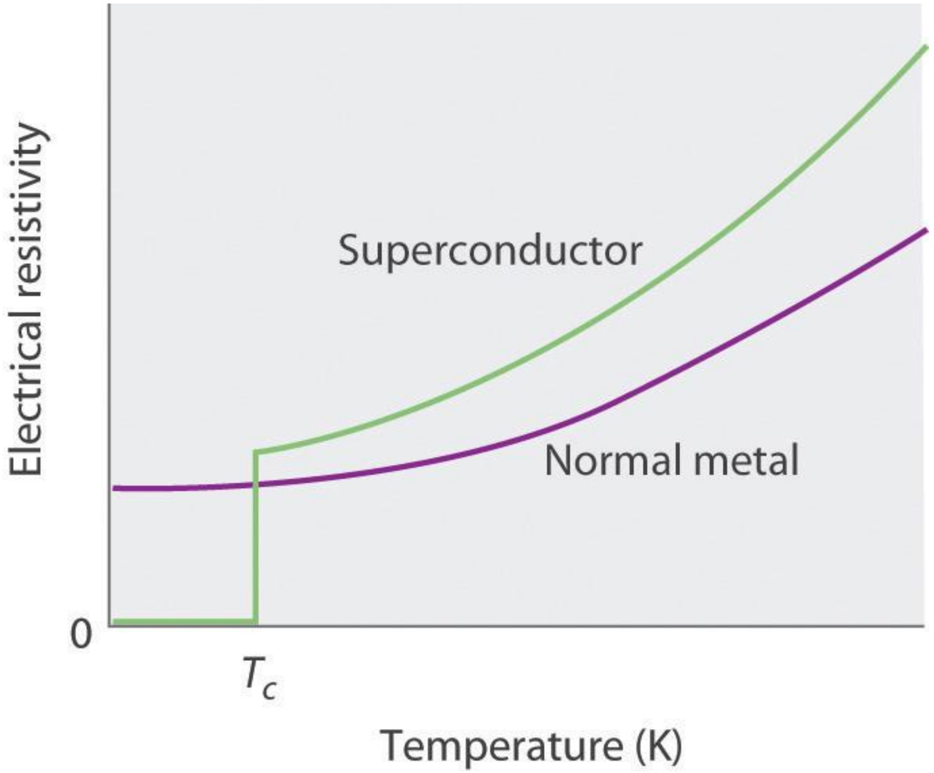
Let’s start again from the concepts

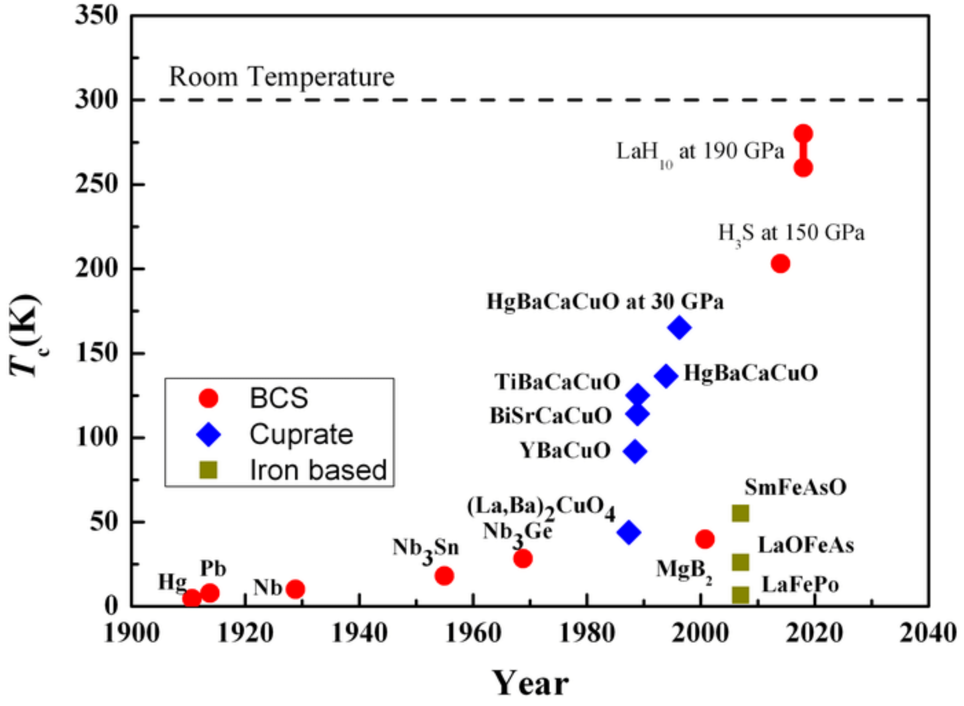
of order and rigidity….

Superconductivity in a nutshell

"superconductor" | T_c | "metal"

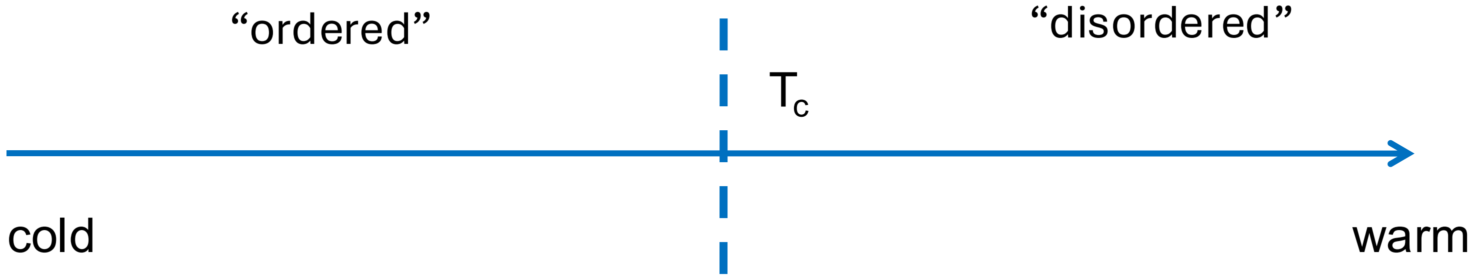
cold → warm



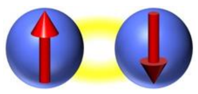


Which symmetry is broken? What is rigid?

Superconductivity in a nutshell

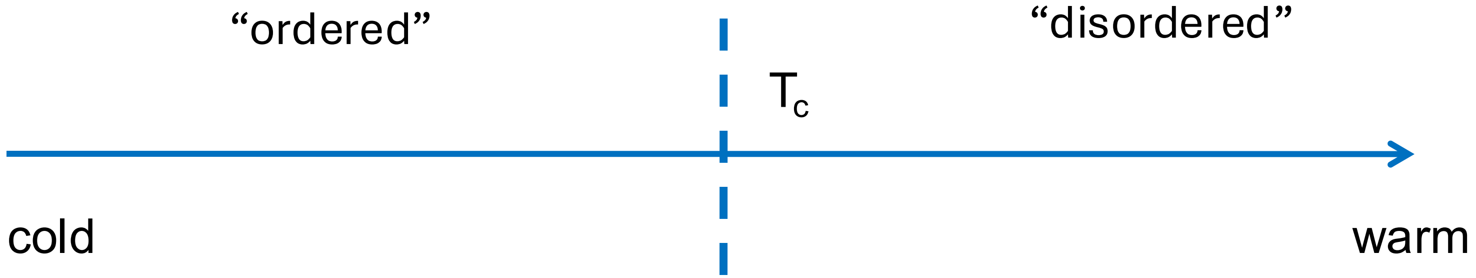


A macroscopic phase-coherent electronic state where electrons with opposite momentum pair up



$$\psi = \langle c_{\downarrow} c_{\uparrow} \rangle = \Delta_0 e^{i\theta_0}$$

Superconductivity in a nutshell



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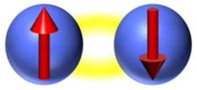

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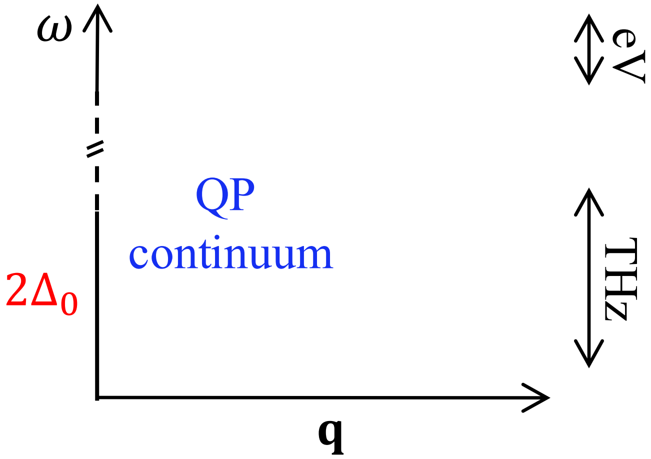
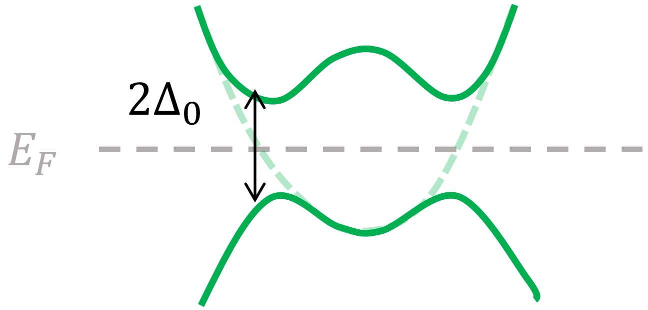
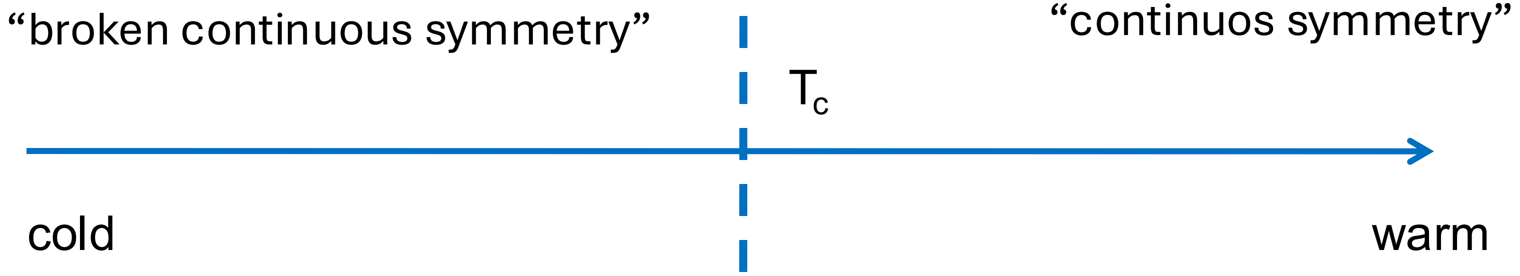
Electronic spectrum develops a “gap”





$$E_k = \sqrt{\xi_k^2 + \Delta_0^2}$$

Superconductivity in a nutshell



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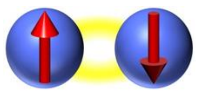
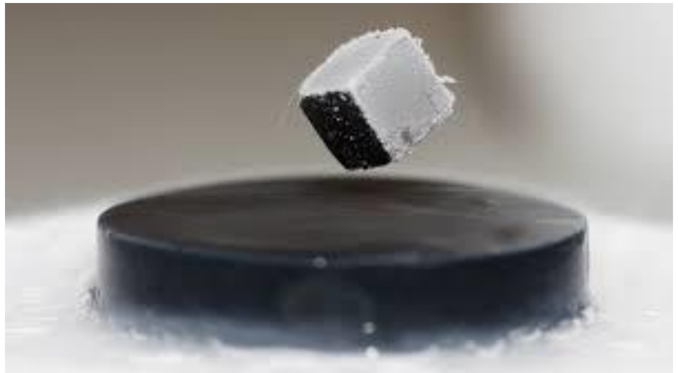
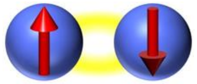


$$\psi = \langle c_\downarrow c_\uparrow \rangle = \Delta_0 e^{i\theta_0}$$

Meissner effect and expulsion of magnetic field: the quantum phase of electrons becomes “rigid”. U(1) global invariance is broken



Ginzburg-Landau Theory



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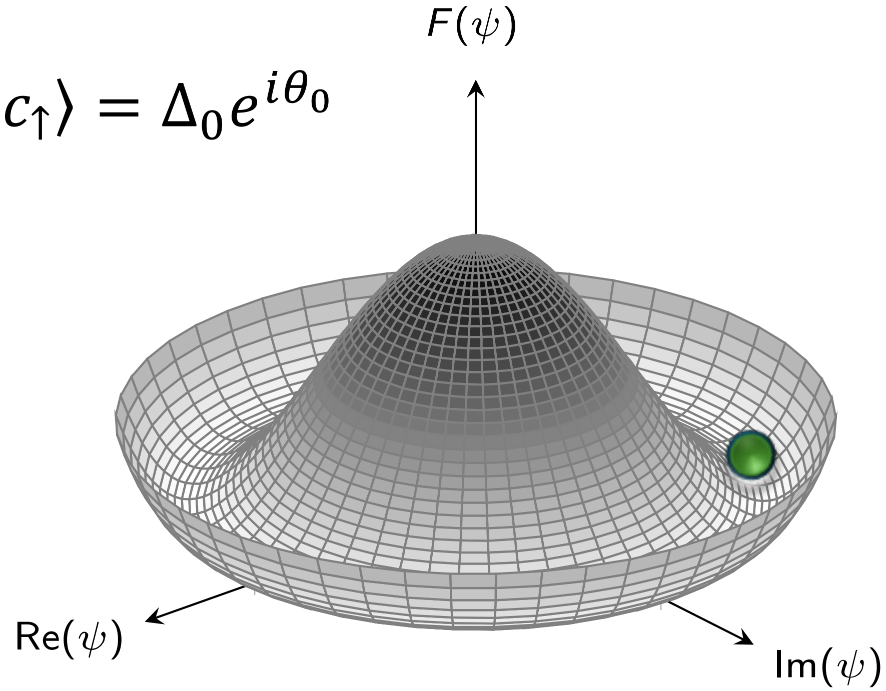


$$\alpha = A(T - T_c)$$

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{4m}|\nabla\psi|^2$$

$$|\psi_0| = \Delta_0 = -\frac{\alpha}{\beta}$$

U(1) symmetry, charge conservation and gauge invariance

- Invariance of the Hamiltonian under global $U(1)$ phase change

$$U(1):\ \ \psi(x) \Rightarrow \psi'(x) = \psi(x)e^{i\alpha} \qquad\qquad \alpha = constant$$

- Whenever interaction V is written in terms of density-density interactions H is invariant under global $U(1)$

$$H = K + V \qquad K = \int dx\, \psi^{+}(x)\frac{(-i\hbar\nabla)^2}{2m}\psi(x) \qquad V = \int dx\, dx'\psi^{+}(x)\psi^{+}(x')V(x-x')\psi(x)\psi(x')$$

- What is the related conserved quantity? (see Emmy Noether's theorem)

- The operator which performs the phase change α on the field operator is:

$$U = e^{i\alpha N} \qquad\qquad N = \int dx\, \psi^{+}(x)\psi(x)$$

- As a consequence, the invariance of H implies that:

$$UHU^{+} = H \Rightarrow \ [H, U] = 0$$

Symmetry under a global phase change leads to charge (density) conservation

U(1) symmetry, charge conservation and gauge invariance

- An Hamiltonian invariant under a global $U(1)$ phase change becomes invariant under a local phase change $\alpha(x)$ by minimal coupling field operators to a gauge field:

$$K = \frac{1}{2m}\int dx\, \psi^{+}(x)\left(-i\hbar\nabla - \frac{qA}{c}\right)^{2}\psi(x)$$

- Now the gauge transformation involves both the field operators and the gauge field:

$$A(x) \rightarrow A(x) + \frac{\hbar c}{q}\nabla\alpha(x)$$
$$\psi(x) \rightarrow \psi(x)e^{i\alpha(x)}$$

- The invariance of the coupled matter-e.m. field model does not lead to any additional conserved quantity (in this sense sometimes it is said that gauge invariance is related to charge conservation)

"superconductor of any kind is nothing more or less than a material in which a particular symmetry of the laws of the nature, electromagnetic gauge invariance, is spontaneously broken"

Steven Weinberg, Talk at Urbana celebrating 50th anniversary of BCS (Cern courier, February 2008)

http://cerncourier.com/cws/article/cern/32522

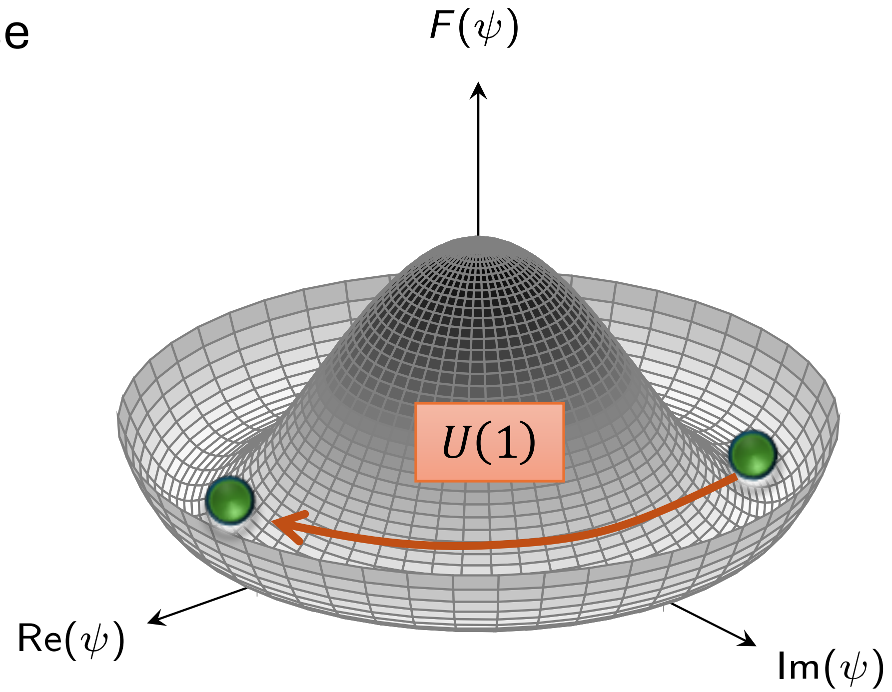
U(1) symmetry, charge conservation and gauge invariance

- The BCS wave function ***is not invariant*** under a global phase change

$$U(1):\ \ c_{k\sigma} \Rightarrow c_{k\sigma}e^{i\alpha} \qquad |\psi_{BCS}\rangle = \prod_k \left(u_k + v_k c_{k\uparrow}^{+} c_{-k\downarrow}^{+}\right)|0\rangle \quad \Rightarrow \quad |\psi_{BCS}\rangle = \prod_k \left(u_k + v_k e^{-2i\alpha} c_{k\uparrow}^{+} c_{-k\downarrow}^{+}\right)|0\rangle$$

- But simultaneously also the order parameter changes by a phase

$$\Delta \Rightarrow \Delta e^{2i\alpha}$$



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- But simultaneously also the order parameter changes by a phase

$$\Delta \Rightarrow \Delta e^{2i\alpha}$$

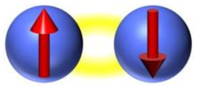
- Remark: the electromagnetic response function computed within the BCS approximation (only Green's functions are corrected, no vertex corrections) violates gauge invariance

$$j^\mu = K^{\mu\nu}(k,\omega)A_\nu \qquad K^{xx}\big(k_x \to 0, k_y = k_z = 0, \omega\big) = 0 \qquad \boxed{K_{BCS}^{xx}\big(k_x \to 0, k_y = k_z = 0, \omega\big) \neq 0}$$

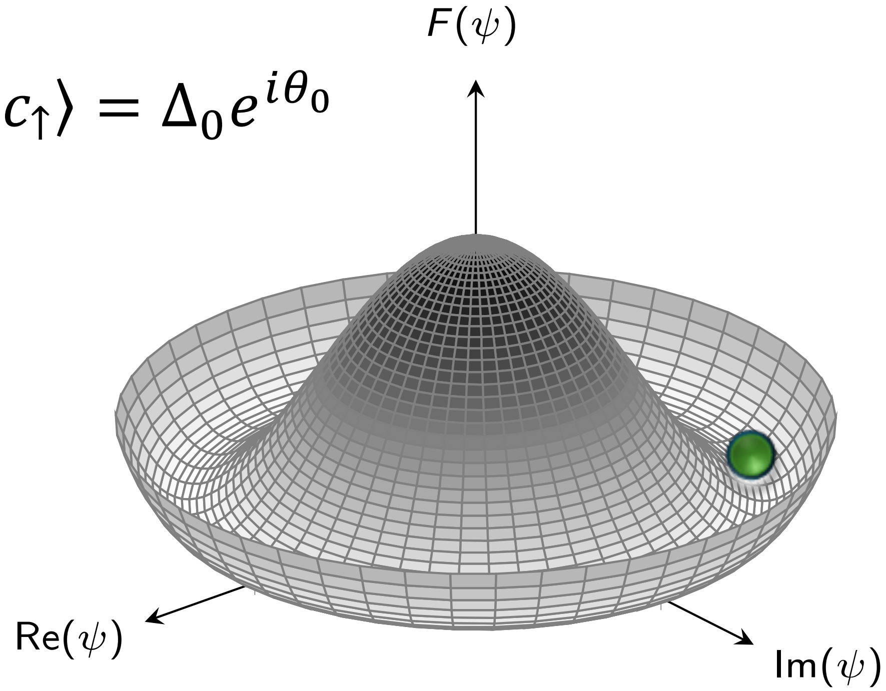
Phase fluctuations must be added to $K_{BCS}^{\mu\nu}$ to restore gauge invariance of the electromagnetic response in the superconductor

Schrieffer's book
Paramekanti et al. PRB **62**, 6786 (2000)
Benfatto et al. PRB 69, 184510 (2004)

Ginzburg-Landau Theory



$$\psi = \langle c_{\downarrow} c_{\uparrow} \rangle = \Delta_0 e^{i\theta_0}$$

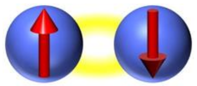


$$\alpha = A(T - T_c)$$

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{4m}|\nabla\psi|^2$$

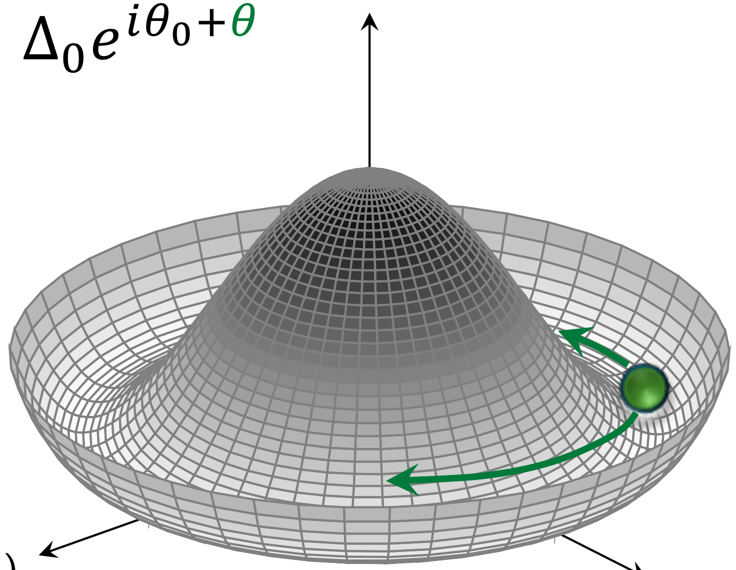
$$|\psi_0| = \Delta_0 = -\frac{\alpha}{\beta}$$

Ginzburg-Landau Theory



$$\psi = \langle c_{\downarrow} c_{\uparrow} \rangle = \Delta_0 e^{i\theta_0 + \theta}$$

Goldstone mode
"massless"



Phase stiffness is like elastic energy in a solid body

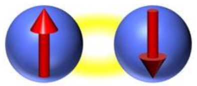
$$\alpha = A(T - T_c)$$

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{4m}|\nabla\psi|^2 \Rightarrow \delta F = \frac{\hbar^2 n_s}{8m}\int dx(\nabla\theta)^2 = \frac{\hbar^2 n_s}{8m}\int dk\, k^2|\theta(k)|^2$$

$$|\psi_0| = \Delta_0 = -\frac{\alpha}{\beta}$$

$n_s \sim |\psi_0|^2$ *superfluid density*

Anderson-Higgs mechanism



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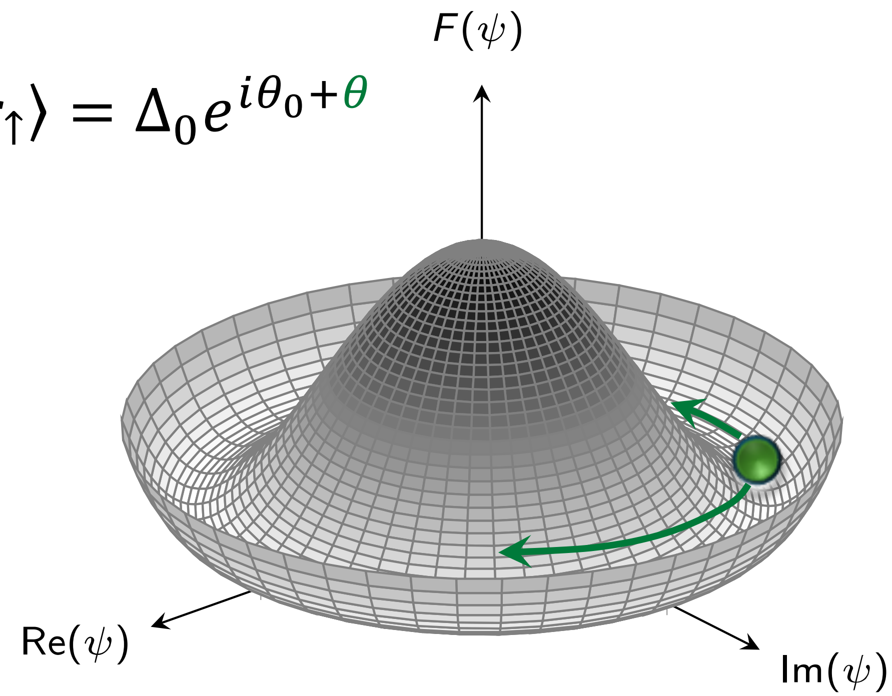

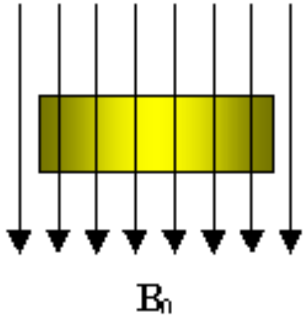

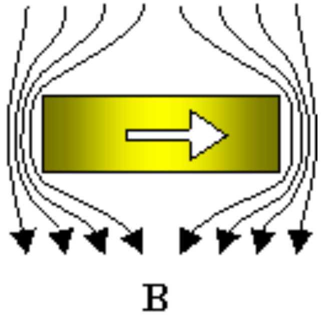

Phase stiffness is like elastic energy in a solid body

$$\delta F = \frac{1}{8\pi} \int dx \left[\frac{4\pi e^2 n_s}{mc^2} A^2 + (\nabla \times A)^2 \right] \Leftarrow \delta F = \frac{n_s}{8m} \int dx \left(\hbar \nabla \theta - \frac{2eA}{c} \right)^2 + \frac{(\nabla \times A)^2}{8\pi}$$

Meissner effect

$$\boxed{\frac{1}{\lambda^2} A^2 = \nabla^2 A}$$

$$\boxed{\frac{1}{\lambda^2} = \frac{4\pi e^2 n_s}{mc^2}}$$

London penetration depth

XY model as a 2D effective phase model

- Mapping between fluctuations of the SC phase and fluctuations of the angular variable in the XY model

$$\delta F = \frac{\hbar^2 n_s^{2D}}{8m} \int dx (\nabla\theta)^2$$

$$H_{XY} = -J \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \simeq \frac{J}{2} \int d\mathbf{r} [\nabla\theta(\mathbf{r})]^2$$

$$J = \frac{\hbar^2 n_s^{2D}}{4m}$$

$$\mathbf{S}_i = S e^{i\theta_i}$$

Classical XY spins

$$\psi_i = \Delta e^{i\theta_i}$$

Charged Superconductor

$$\psi_i = \rho e^{i\theta_i}$$

Neutral Superfluid

Here we neglect ***amplitude fluctuations*** of the order parameter

XY model as a 2D effective phase model

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- Which signatures can we expect?

$$J_s(T_{KT}^-) = \frac{2T_{KT}}{\pi}, \quad J_s(T_{KT}^+) = 0$$

"Universal" BKT jump of the stiffness

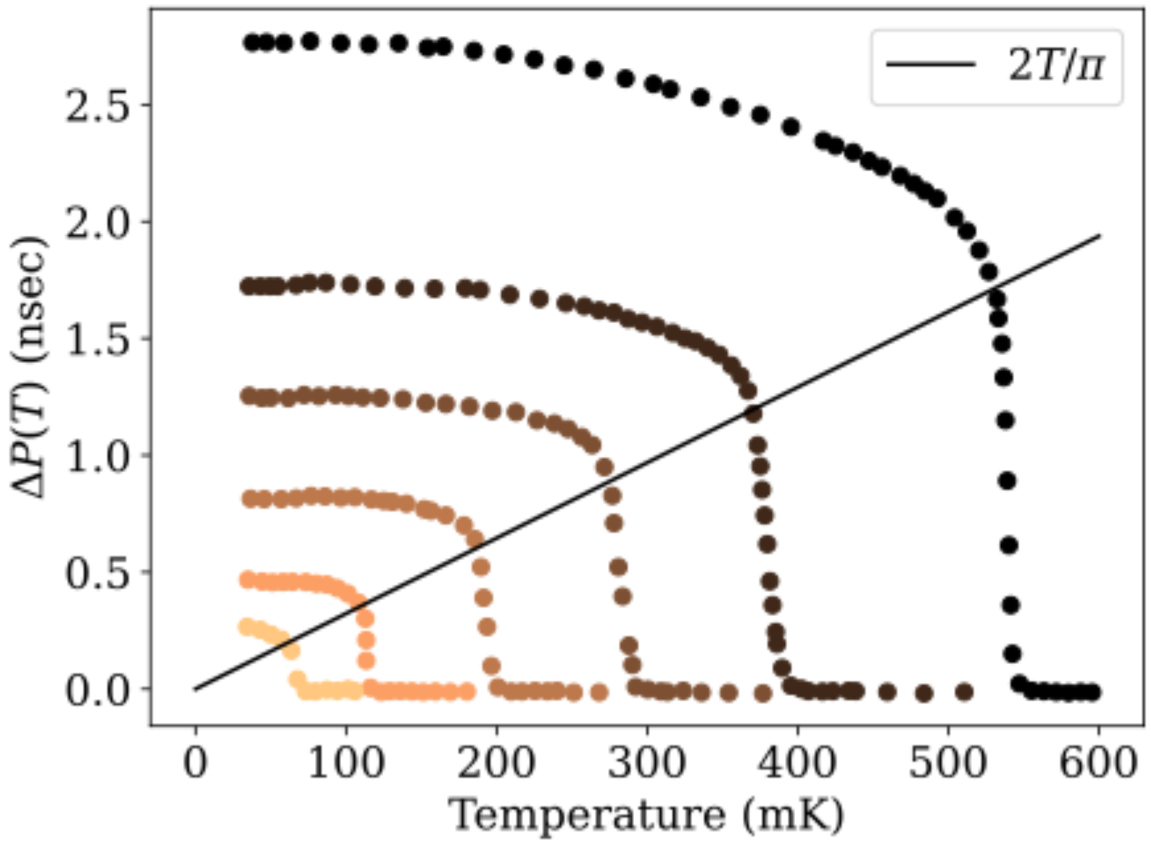
Exponential divergence of the correlation length

$$\xi_{BKT} = a e^{b/\sqrt{t}}$$

These features must be properly connected to experimentally accessible quantities

Neutral superfluid

- Oscillation period of a torsional pendulum immersed in liquid Helium: when a fraction of the fluid is irrotational the oscillation period changes



Bishop & Reppy, PRL 78, PRL 84

$$\Delta P(T) \propto J_s = \frac{\hbar^2 n_s^{2D}}{4m}$$

$$J_s(T_{KT}^-) = \frac{2T_{KT}}{\pi}, \quad J_s(T_{KT}^+) = 0$$

J. Phys. C: Solid State Phys., Vol. 5, 1972.

Long range order and metastability in two dimensional solids and superfluids

J M KOSTERLITZ and D J THOULESS

Abstract. Dislocation theory is used to define long range order for two dimensional solids. An ordered state exists at low temperatures, and the rigidity modulus is nonzero at the transition temperature. Similar arguments show that the superfluid density is nonzero at the transition temperature of a two dimensional superfluid.

J. Phys. C: Solid State Phys., Vol. 6, 1973.

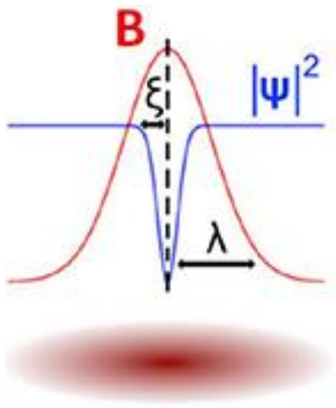
Ordering, metastability and phase transitions in two-dimensional systems

J M Kosterlitz and D J Thouless

Abstract. A new definition of order called topological order is proposed for two-dimensional systems in which no long-range order of the conventional type exists. The possibility of a phase transition characterized by a change in the response of the system to an external perturbation is discussed in the context of a mean field type of approximation. The critical behaviour found in this model displays very weak singularities. The application of these ideas to the xy model of magnetism, the solid–liquid transition, and the neutral superfluid are discussed. This type of phase transition cannot occur in a superconductor nor in a Heisenberg ferromagnet, for reasons that are given.

Interaction between vortices in a *charged* superfluid

$$\delta F = \frac{n_s}{8m}\left(\hbar\nabla\theta - \frac{2e}{c}A\right)^2 + \frac{(\nabla \times A)^2}{8\pi}$$



- A finite $\nabla\theta$ generates a finite A, which screens current out

$$I = -\frac{\partial F}{\partial A} = \frac{e n_s}{2mc}\left(\hbar\nabla\theta - \frac{2e}{c}A\right)$$

- In a charged superfluid interaction between vortices is logarithmic only up to a scale λ $\quad \lambda^2 = \frac{mc^2}{4\pi e^2 n_s}$
- However, three crucial observations make BKT physics possible in thin films of superconductors:
  - In a film of thickness d the cut-off scale is the so-called Pearl length (Pearl, 1964)

$$\boxed{\Lambda = \frac{\lambda^2}{d}}$$

  - In thin films disorder further suppress n_s as compared to bulk, making λ larger
  - ***Quasiparticle excitations*** suppress $\Lambda(T)$ as T approaches the BCS temperature T_c

Λ can be as large as cm near to T_c

Beaesly, Mooji, Orlando PRL79, Halperin and Nelson JLTP 79, Hebard & Fiory 82

The BKT signatures in superconducting films

- From below: “universal” jump of the stiffness
  - direct measurements of the inverse penetration depth
  - measurements of the I-V characteristics

- From above: exponential divergence of the correlation length
  - measurements of the resistivity
  - measurements of the magnetization

In all these cases one should be extremely careful in identifying the correct temperature regime where BKT physics can occur

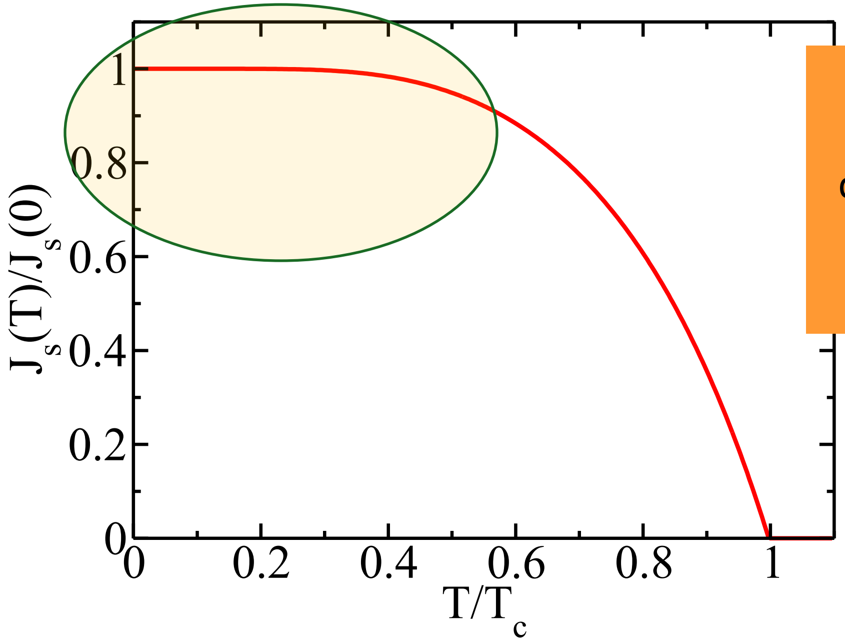
In real systems many phenomena can conspire to mask BKT signatures

The BKT signatures by approaching the transition from below T_{BKT}:
the universal jump of the superfluid density

Direct measurements of $n_s \propto \lambda^{-2}$ via two-coils mutual unductance techinques
Indirect measurement of n_s via the exponent of IV characteristics

The BKT jump

- Where should we expect to see the BKT jump?



In superconductors J_s is suppressed by quasiparticle excitations

In 3D J_s goes to zero ***continuously*** at T_c

$$J_s(T_{BKT}) = \frac{2T_{BKT}}{\pi}$$

$$\frac{J_s(T_{BKT})}{J_s(T=0)} = \frac{2T_{BKT}}{\pi T_c} \frac{T_c}{J_s(T=0)}$$

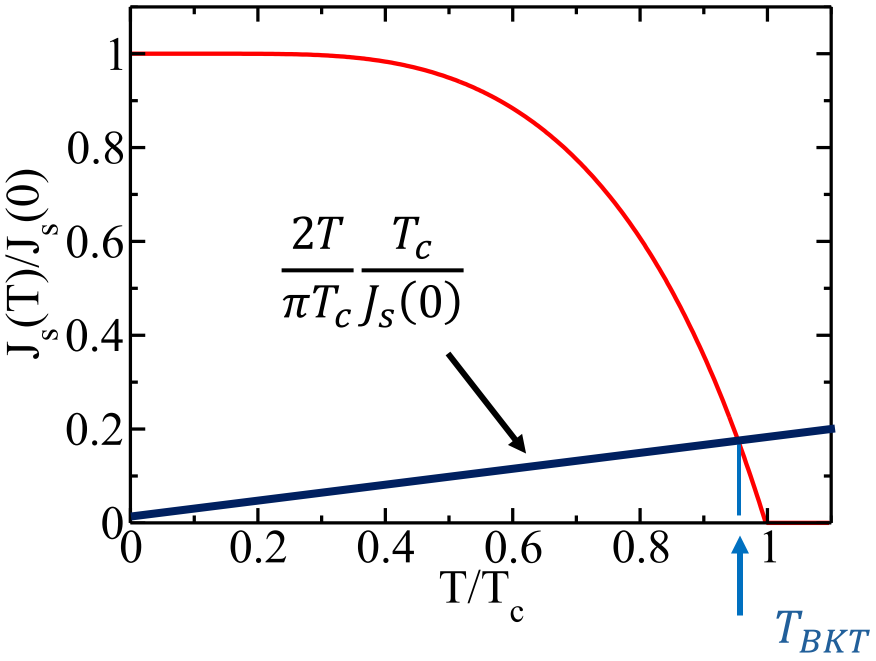
Low-T suppression due to quasiparticle excitations $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$

$$J_s(T) \sim J_s(0) - \alpha e^{-\Delta/k_B T}$$ s-wave

$$J_s(T) \sim J_s(0) - \alpha T$$ d-wave

The BKT jump

- Where should we expect to see the BKT jump?



Slope ~ $T_c / J_s(0)$

$$J_s(T_{BKT}) = \frac{2T_{BKT}}{\pi}$$

$$\frac{J_s(T_{BKT})}{J_s(T=0)} = \frac{2T_{BKT}}{\pi T_c}\frac{T_c}{J_s(T=0)}$$

Conventional superconductors

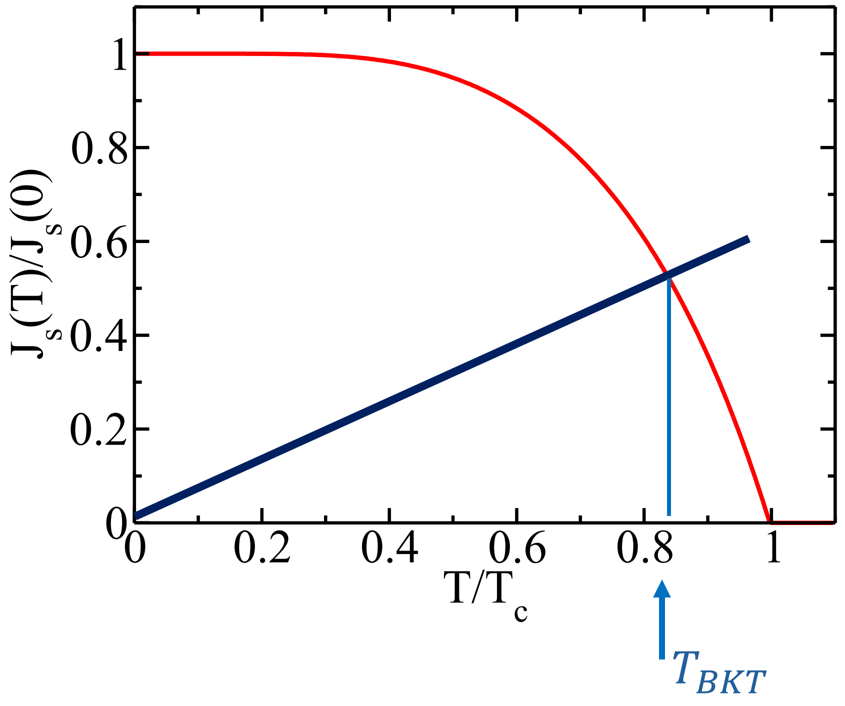
$J_s(0) \sim E_F$

$T_c \ll J_s(0)$

$$T_{BKT} \approx T_c$$

The BKT jump

- Where should we expect to see the BKT jump?



Slope $\sim T_c / J_s(0)$

$$J_s(T_{BKT}) = \frac{2T_{BKT}}{\pi}$$

$$\frac{J_s(T_{BKT})}{J_s(T=0)} = \frac{2T_{BKT}}{\pi T_c} \frac{T_c}{J_s(T=0)}$$

High-temperature superconductors

$J_s(0) \sim T_c$

$$T_{BKT} < T_c$$

In principle easier to detect BKT signatures

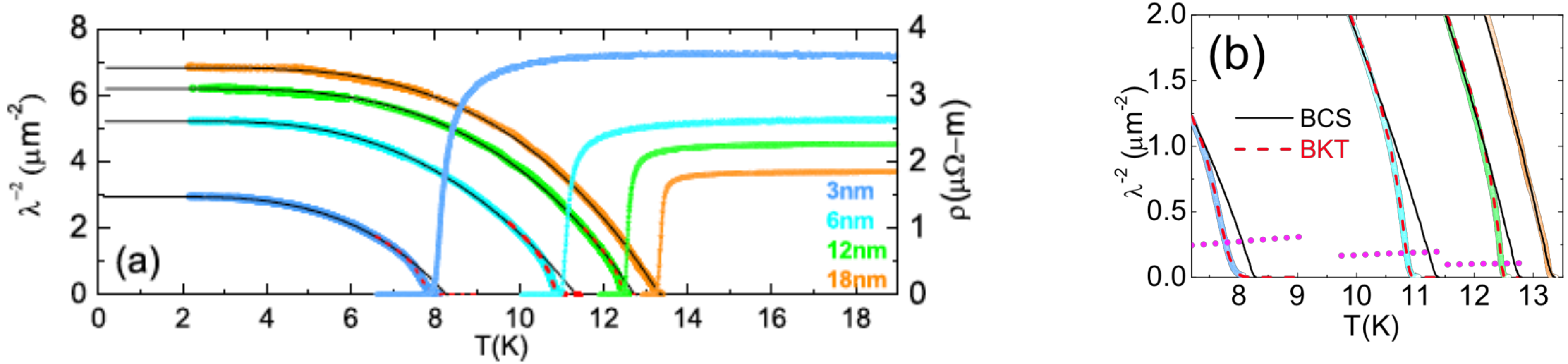
In practice several additional complications in

layered 3D systems with marked ***inhomogeneities***

L. Benfatto, C. Castellani and T. Giamarchi, Phys. Rev. Lett. **98**, 117008 (2007)

BKT jump in conventional NbN thin films

- Very difficult to see the jump because T_{BKT} almost coincides with the BCS critical temperature



M.Mondal, L.B. et al. PRL 107, 217003 (2011)

No real jump, but a quite broad downturn

Effect of ***inhomogeneities***: the global conductivity is an average over different local T^{i}_{BKT}

BKT jump in conventional NbN thin films

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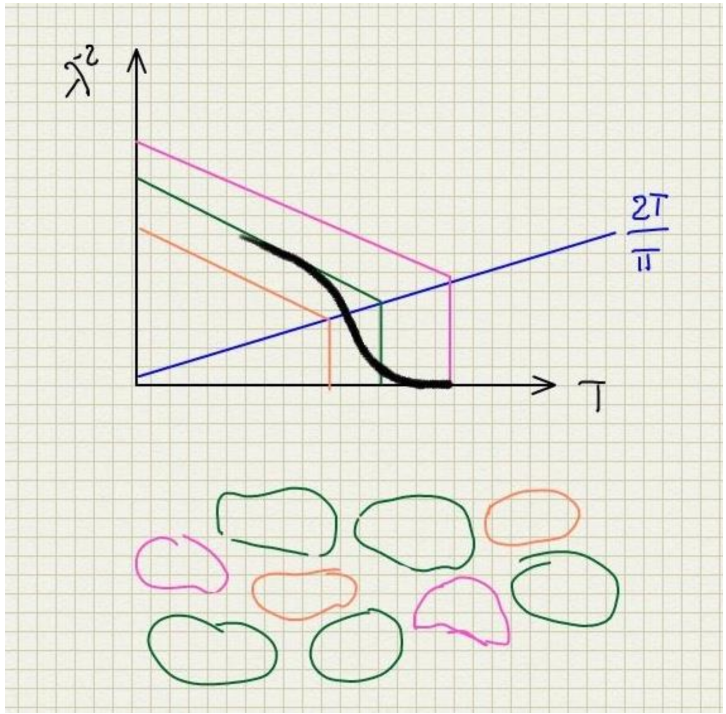
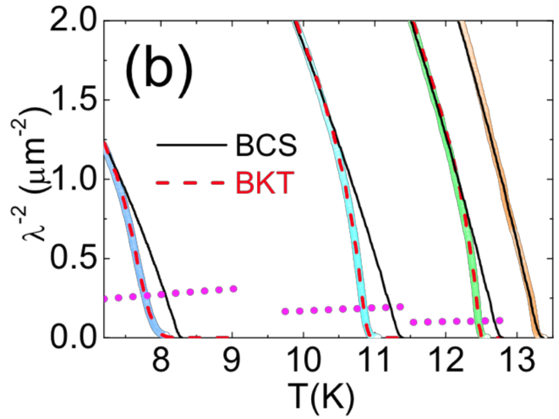




M.Mondal, L.B. et al. PRL 107, 217003 (2011)

Effect of ***inhomogeneities***: the global conductivity is an average over different local T_{BKT}^{i}

$$\bar{\sigma} = \sum_i P_i \sigma_i$$

The intrinsic **inhomogeneity** in the local superconducting properties leads to a **smearing** of the universal jump
(L.B. et al. PRB 2008, PRL 11, PRB 13)

Initially introduced as a phenomenological effect
More recently validated by Monte Carlo simulations

Disorder and superconductivity

superconductivity

disorder/insulator

macroscopic **phase** coherence

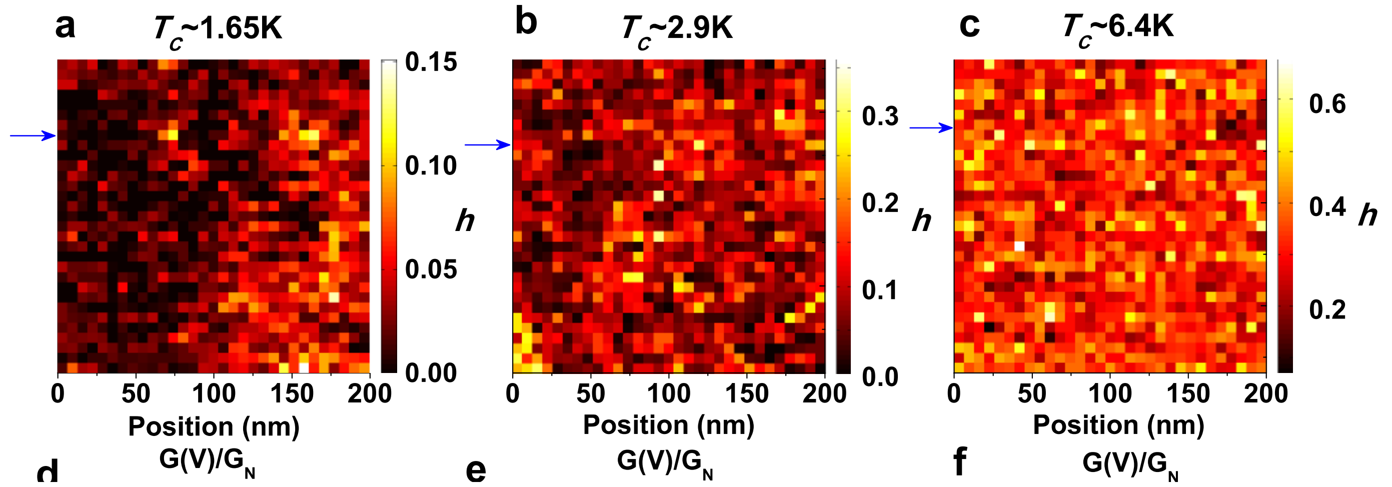
inhomogeneous SC background in real space

macroscopic **charge** localization

Experimentally observed by Scanning Tunneling Microscopy

G. Lemarie, L.B. et al. PRB 87, 184509 (13)

NbN disordered films



Disorder and superconductivity

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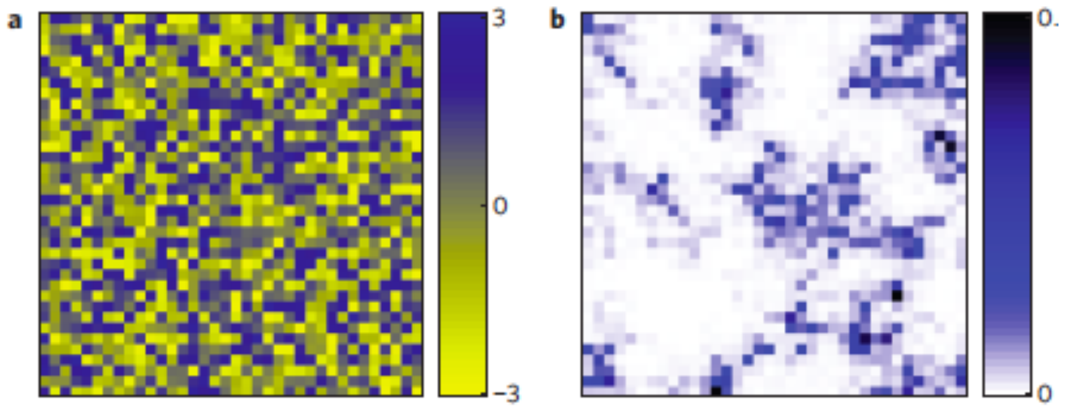
Experimentally observed by Scanning Tunneling Microscopy
Theoretical justified with numerical Bogoliubov-de-Gennes solutions for disordered models

distribution of V_i



distribution of Δ_i

Ghosal et al. PRB 65, 014501 (2001); Bouadim, et al. Nat. Phys. 2011

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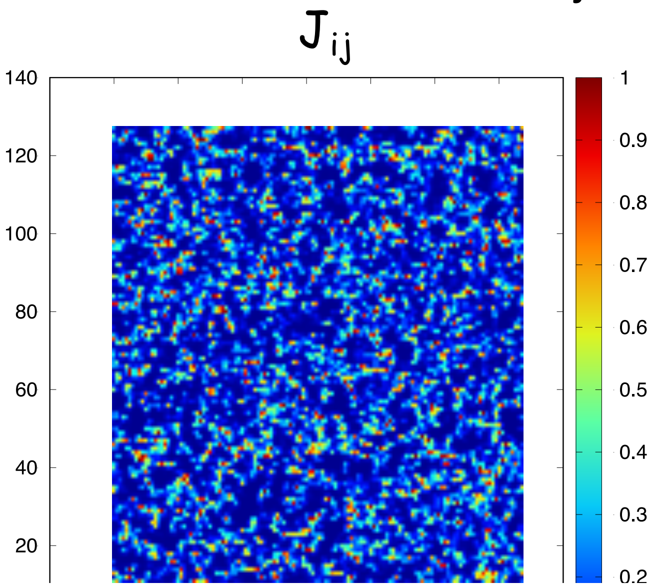


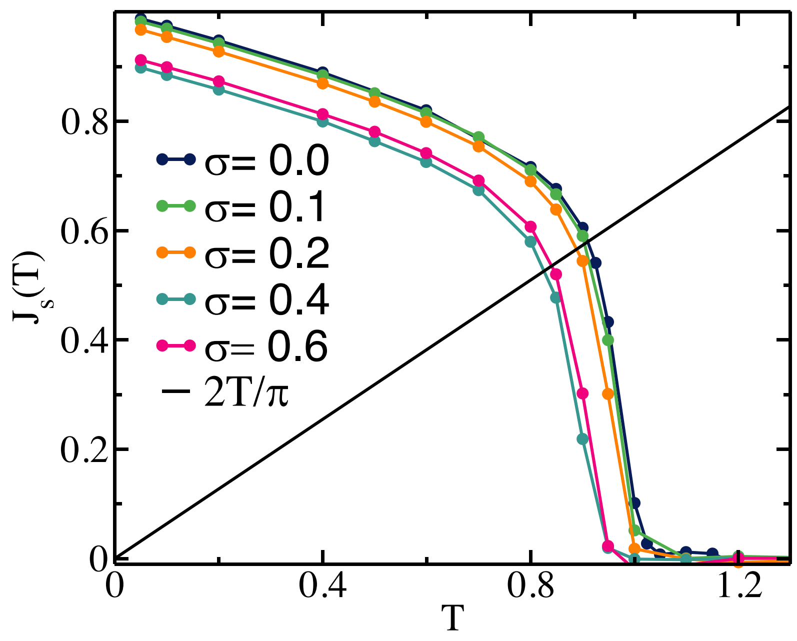
$$H = -\sum_{ij} J_{ij} \cos(\theta_i - \theta_j)$$

What is the effect of correlated disorder of the local couplings on the BKT transition?

BKT transition in inhomogeneous background

- *Uncorrelated* disorder has no effect on the sharp BKT jump

$$H = -\sum_{ij} J_{ij}\cos(\theta_i - \theta_j)$$



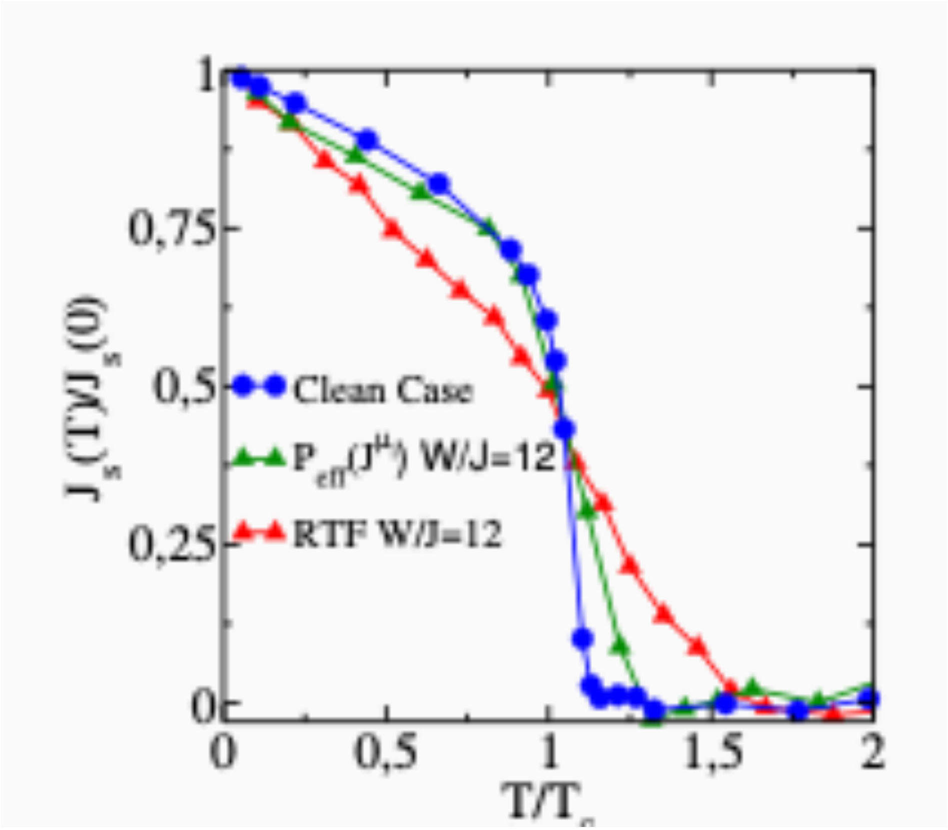
J_{ij} distributed with a Gaussian probability $P(J_{ij})$

Monte Carlo simulations

Ilaria Maccari
presently SNSF fellow at ETH Zurich

BKT transition in inhomogeneous background

- ***Correlated*** disorder broadens considerably the BKT jump

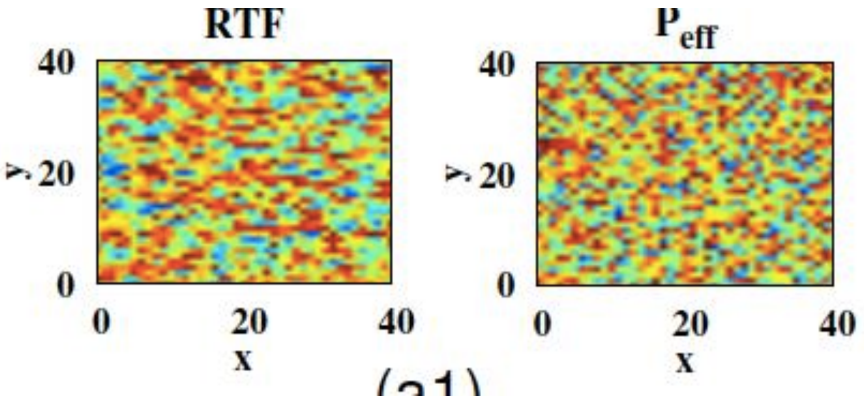


Ilaria Maccari, L.B. and C. Castellani
Phys. Rev B 96, 060508(R) (2017)
Condens. Matter 3, 8 (2018)
Phys. Rev B 99, 104509 (2019)

$$H = -\sum_{ij} J_{ij}\cos(\theta_i - \theta_j)$$

This result justifies the phenomenological view of an inhomogeneity-induced smearing of the transition

$P^{eff}(J_{ij})$: no spatial correlations
RTF: spatially correlated “granular” disorder

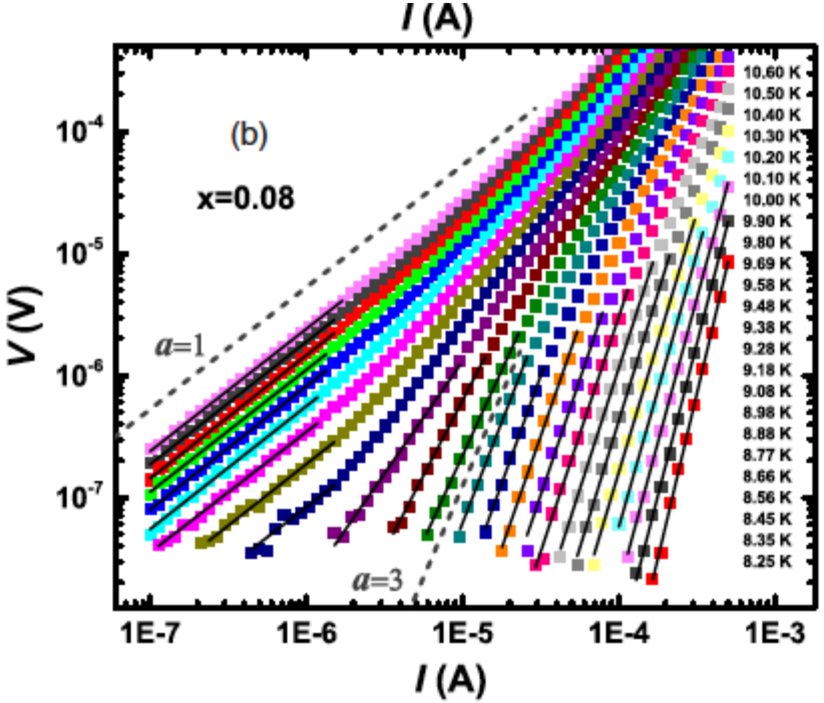


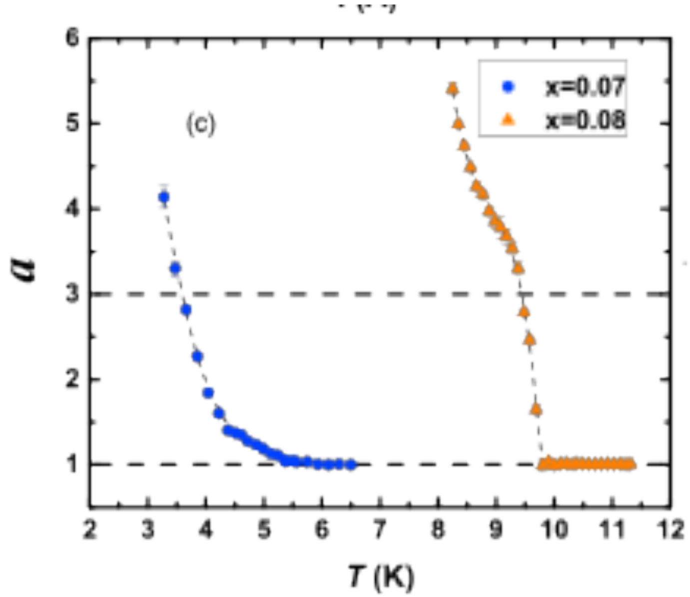
BKT jump from I-V characteristics

- A large enough current can dissociate vortices leading to non-linear I-V characteristics ***below*** T_{BKT}

$$V = n_V I \qquad V = I^{a(T)} \qquad a(T) = \frac{\pi J_s}{T} + 1 \Rightarrow \begin{cases} J_s(T_{BKT}^-) = \frac{2T_{BKT}}{\pi} & a(T_{BKT}^-) = 3 \\ J_s(T_{BKT}^+) = 0. & a(T_{BKT}^+) = 1 \end{cases}$$

Thick films of LSCO





P. G. Baity, Xiaoyan Shi, Zhenzhong Shi, L. Benfatto, Dragana Popović, Phys. Rev. B 93, 024519 (2016)

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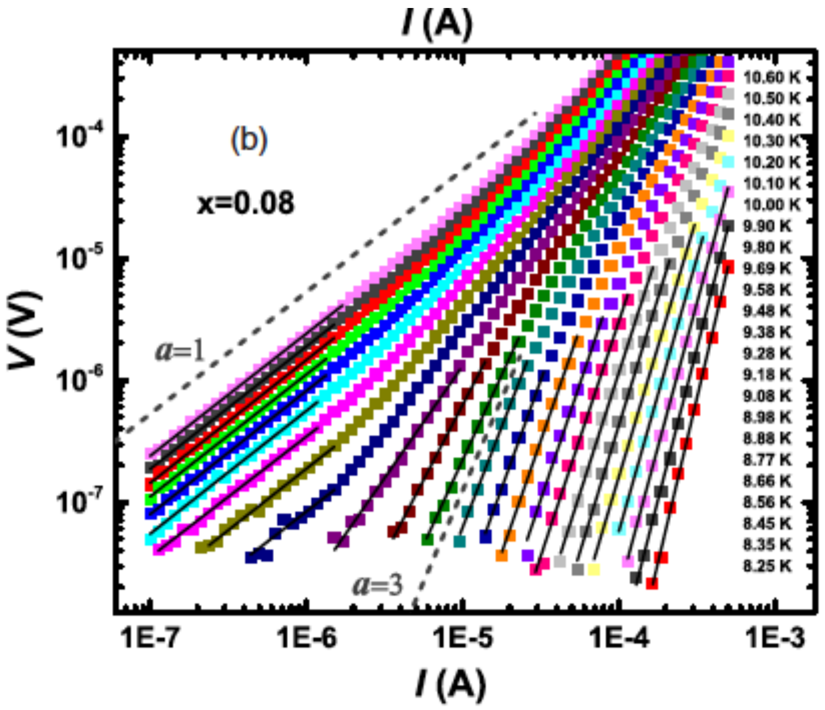
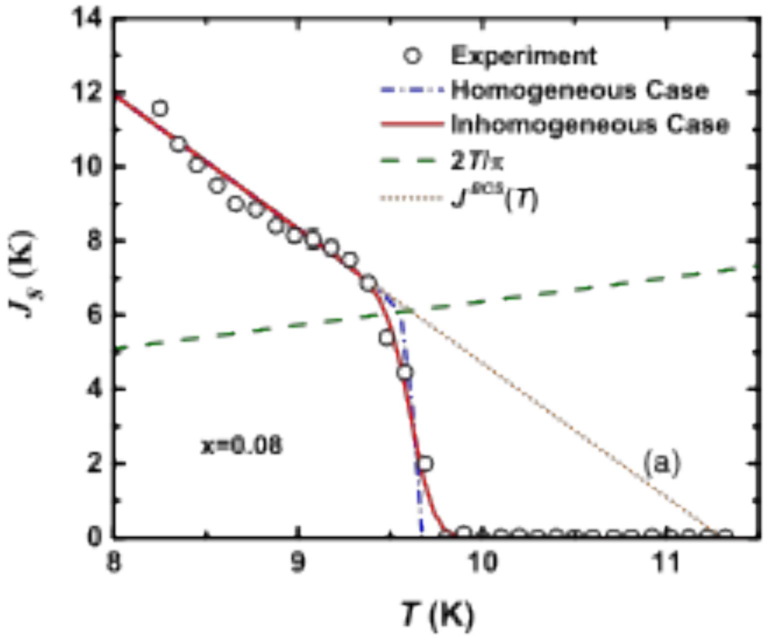


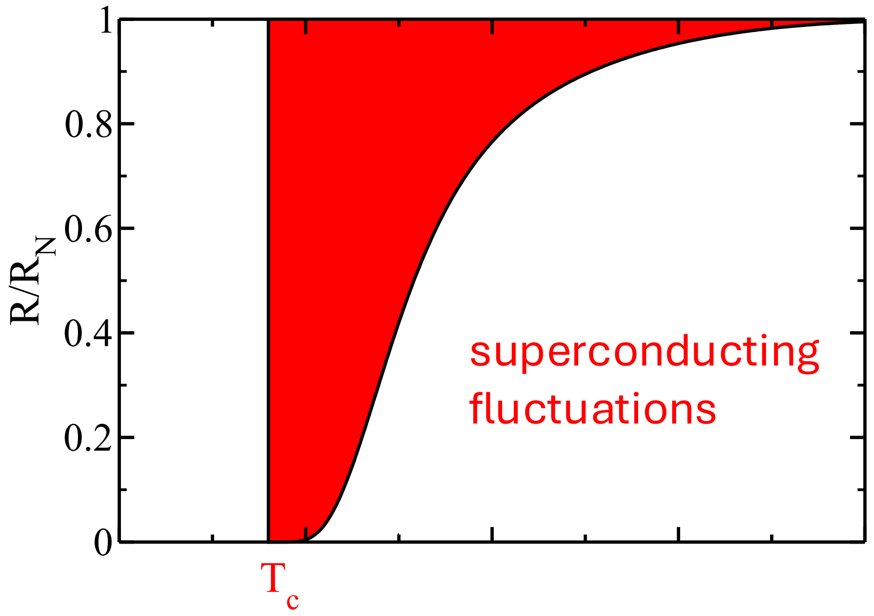


P. G. Baity, Xiaoyan Shi, Zhenzhong Shi, L. Benfatto, Dragana Popović, Phys. Rev. B 93, 024519 (2016)

The BKT signatures by approaching the transition from above T_{BKT}:
the exponential divergence of the correlation length

Paraconductivity: progressive decrease of the resistivity

- Paraconductivity scales as the inverse correlation length, both for Ginzburg-Landau and BKT fluctuations
- However, temperature dependence is markedly different



$$\frac{R}{R_N} = \frac{\xi_0^2}{2\pi\xi^2}$$

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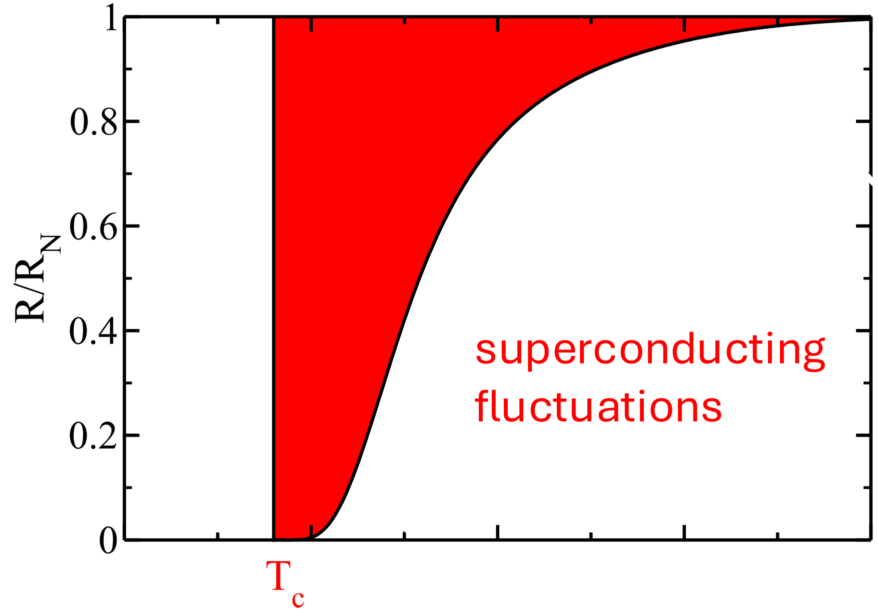


Inhomogeneity can further mask the BKT regime where an exponential behavior can be expected

L. Benfatto, C. Castellani and T. Giamarchi, Phys. Rev. B 80, 214506 (2009)

$$\frac{R}{R_N} = \frac{\xi_0^2}{2\pi\xi^2}$$

Hebard & Fiory 1982

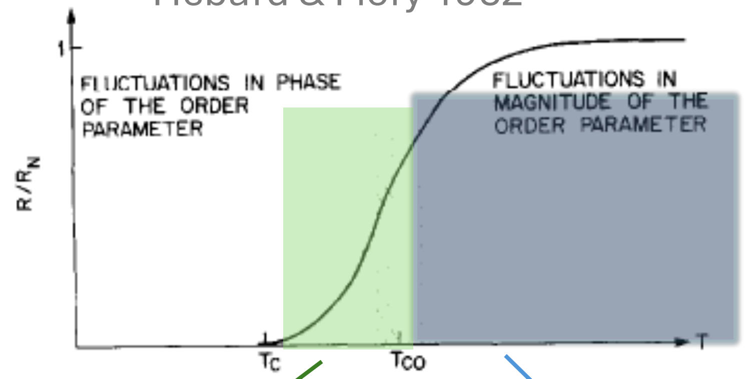


Fig. 1. Typical resistive transition of a 2D superconductor showing mean field temperature T_{co} and 2D phase transition temperature T_c.

$$\xi_{BKT} = a e^{b/\sqrt{t}}$$

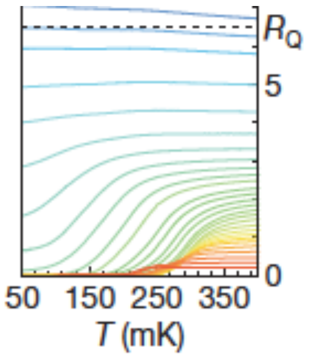
$$t = \frac{T - T_{BKT}}{T_{BKT}}$$

$$\xi_{GL}^2 = \frac{T}{T - T_c}$$

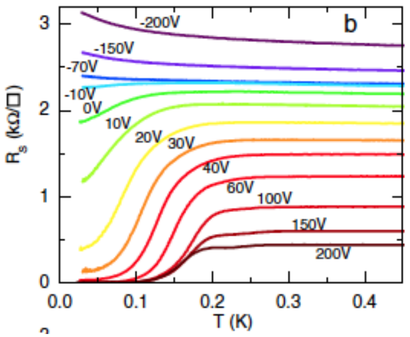
Broad resistive transition in 2D systems

- A very broad resistive transition is very common in "quasi-2D" systems, especially in superconducting interfaces

LAO/STO or LTO/STO Interfaces

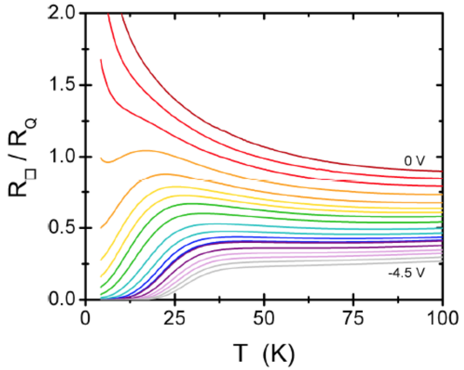


LAO/STO,
A. Caviglia et al. Nature 2008

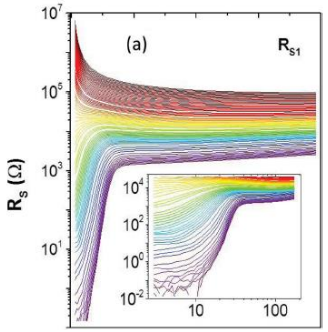


LTO/STO,
J. Biscaras et al. PRL 2012

Cuprates doped via electrolytes

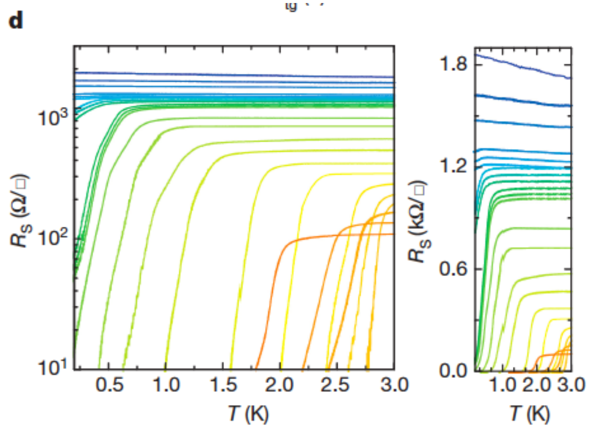


LSCO ultrathin films.
A.T. Bollinger et al. Nature 2011

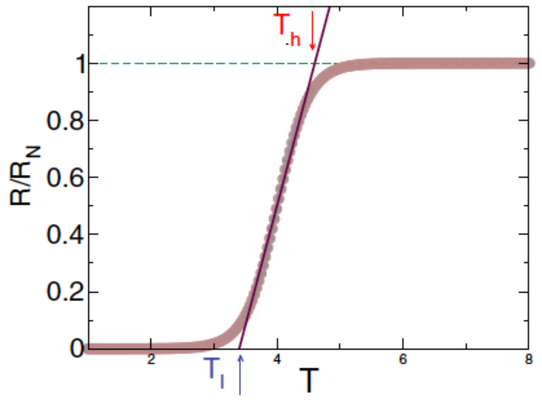


LCO thin films.
J.Garcia-Barriocanal, PRB13

SC-CDW films 1T-TiSe2



L.J. Li et al. Nature 2016



$$\frac{\Delta T_c}{T_c} \sim 1$$

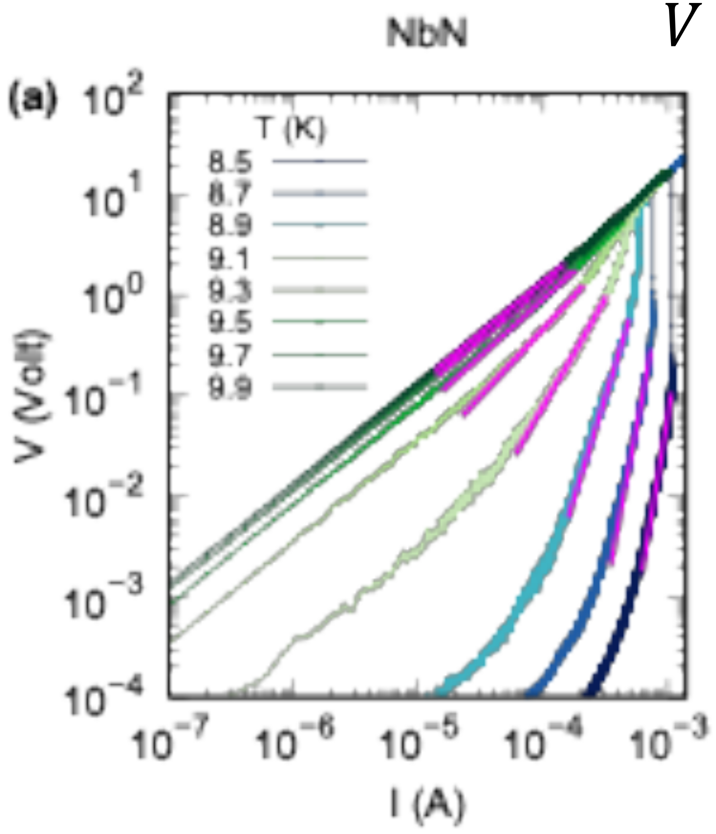
Such a broad transition cannot be captured by conventional fluctuations mechanisms and should be rather ascribed to ***mesoscopic*** inhomogeneity

L. B., C. Castellani and T. Giamarchi, PRB 80, 214506 (2009)
S. Caprara, L.B. et al. PRB 84, 014514 (2011)
R. Ganguly, D. Chaudhuri, P. Raychaudhuri, L. B., PRB 91, 054514 (2015)
G. Singh, A. Jouan, L. B. et al. Nat. Comm. 9, 407 (2018)

Real BKT signatures vs mesoscopic inhomogeneity

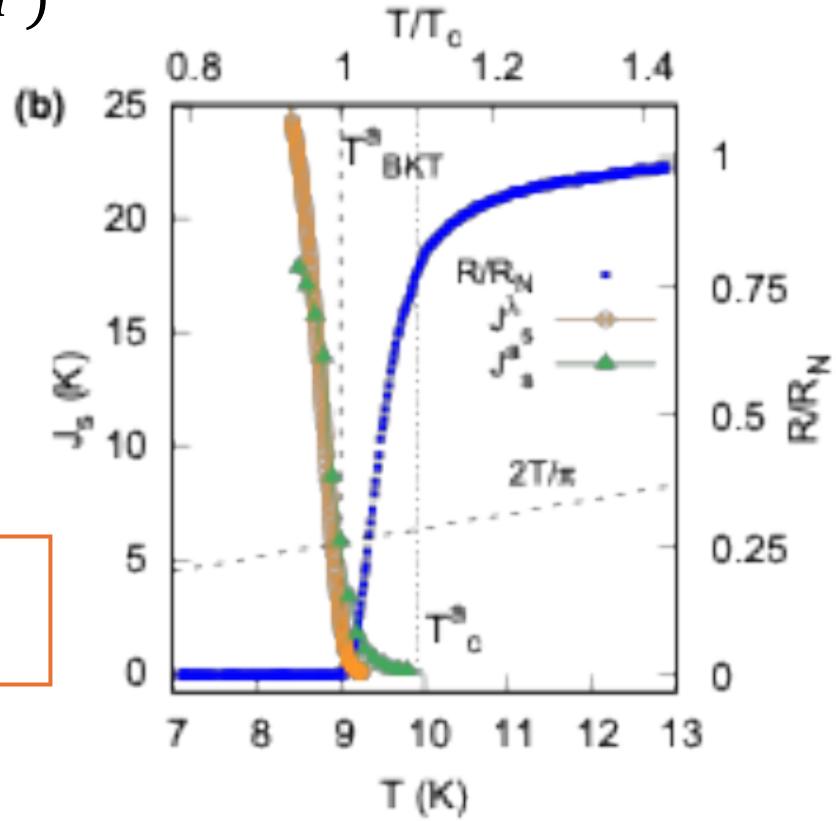
NbB thin films

$$V = I^{a(T)} \quad a(T) = \frac{\pi J_s}{T} + 1 \Rightarrow J_s(T)$$



$J_s(T)$ extracted from IV characteristics in a regime of temperatures ***below*** the one where resistivity vanishes

Consistent description of the resistive and superfluid transition

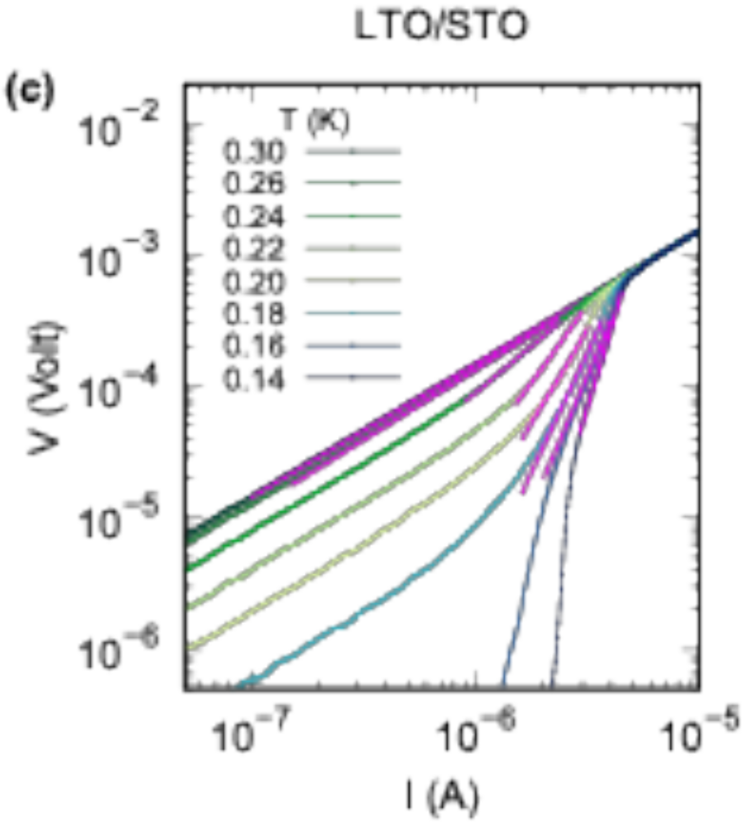


G. Venditti, J. Biscaras, S. Hurand, N. Bergeal, J. Lesueur, A. Dogra, R. C. Budhani, Mintu Mondal, John Jesudasan, Pratap Raychaudhuri, S. Caprara, and L. Benfatto, PRB 100, 064506 (2019)

Real BKT signatures vs mesoscopic inhomogeneity

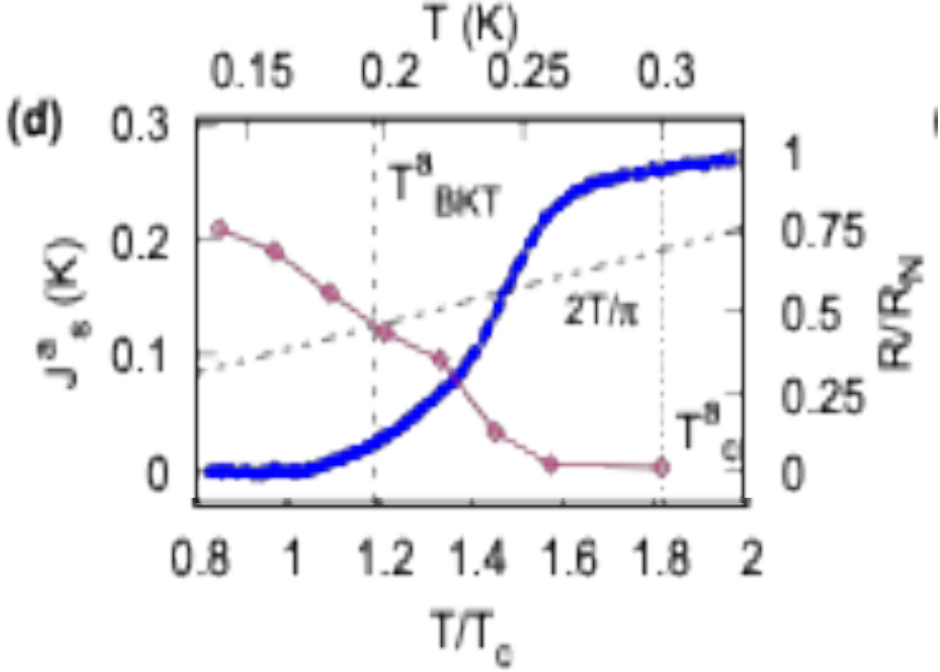
LAO/STO inerfaces

$$V = I^{a(T)} \quad a(T) = \frac{\pi J_s}{T} + 1 \Rightarrow J_s(T)$$



$J_s(T)$ extracted from IV characteristics in the ***wrong*** temperature regime

Inconsistent description of the resistive and superfluid transition (very common in the literature…)

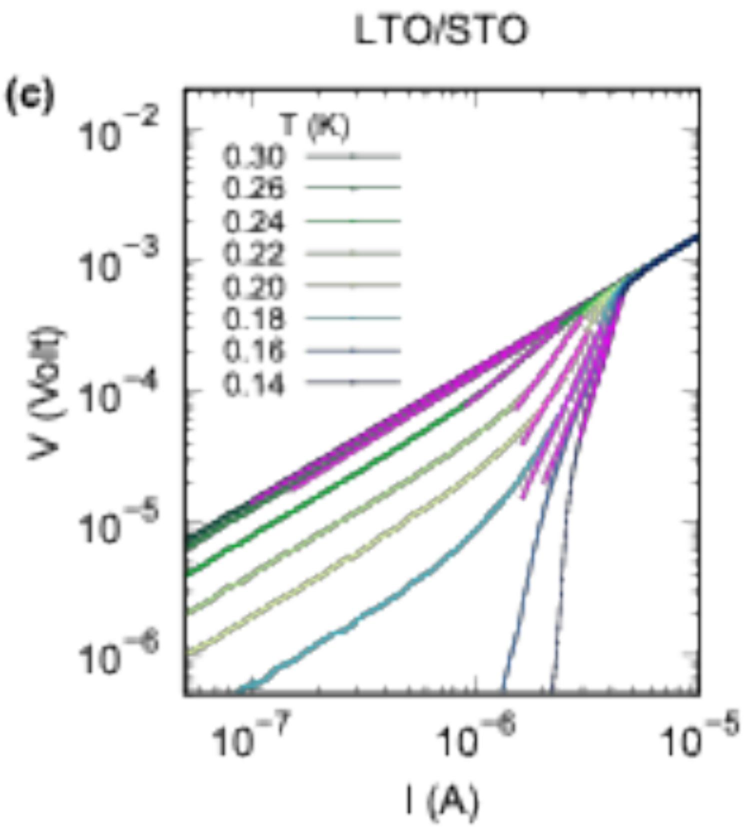


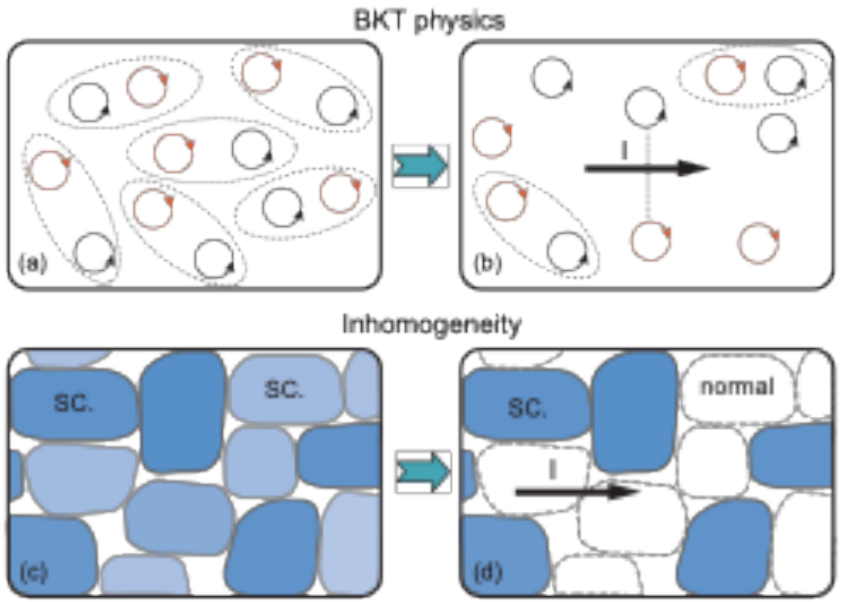
G. Venditti, J. Biscaras, S. Hurand, N. Bergeal, J. Lesueur, A. Dogra, R. C. Budhani, Mintu Mondal, John Jesudasan, Pratap Raychaudhuri, S. Caprara, and L. Benfatto, PRB 100, 064506 (2019)

Real BKT signatures vs mesoscopic inhomogeneity

LAO/STO inerfaces

$$V = I^{a(T)} \quad \cancel{a(T) = \frac{\pi J_s}{T} + 1} \Rightarrow J_s(T)$$





IV-characteristics nonlinearity explained by a different mechanism: progressive local breaking of pairing by current larger than the critical one

G. Venditti, J. Biscaras, S. Hurand, N. Bergeal, J. Lesueur, A. Dogra, R. C. Budhani, Mintu Mondal, John Jesudasan, Pratap Raychaudhuri, S. Caprara, and L. Benfatto, PRB 100, 064506 (2019)

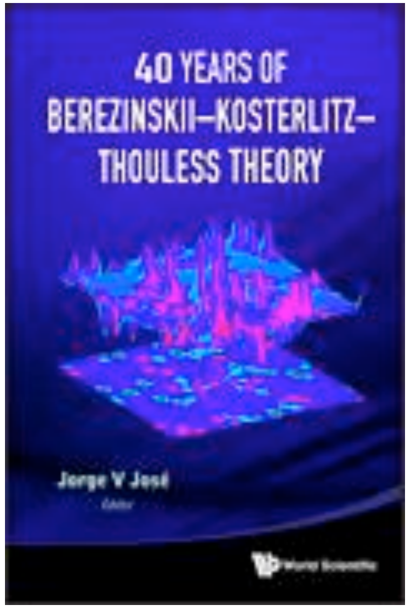
Conclusions

- The BKT transition represents a breakthrough in our view of phase transitions. What really matters is not *order* but ***rigidity***. It manifests in the long-distance behavior of the correlation functions
- Milestone description given by Beresinskii, Kosterlitz and Thouless within the ***classical XY model***
- Its universality class applies to a plethora of phenomena, the most studied one being the superfluid and superconducting transition in the quasi-2D limit
- ***Reality is not the XY model***: BKT signatures in real superconducting films can be affected by several non-universal ingredients: quasiparticle excitations, vortex-core energy, layered structure, etc.
- ***Inhomogeneity*** is a wider concept as disorder, and it turns out to be intimately connected with the observation of BKT physics in quasi 2D superconductors
- BKT theory provides you with a set of beautiful analytical results: but please be careful when you try to fit experimental data with them!

Further reading

- N. Nagaosa «Quantum field theory in condensed matter Physics»
- P. Coleman «Introduction to Many-Body Physics»

- L. Benfatto, C.Castellani and T. Giamarchi Book chapter in «40 years of Beresinskii-Kosterlitz-Thouless theory» edited by Jorge V. José, World Scientific (2013) and arXiv:1201.2307



- Ilaria Maccari, PhD Thesis, Uniroma1

https://iris.uniroma1.it/handle/11573/1719098