Particle-Hole Symmetries in Condensed Matter

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What's a symmetry in quantum mechanics? (I)

- quantum Hilbert space V with Hermitian scalar product $\langle \cdot , \cdot \rangle$.
- quantum state / density matrix : $\varrho = \varrho^\dagger$, $0 \leqslant \varrho \leqslant 1$, $\mathrm{Tr}_V \varrho = 1$.

(explicitly: $\varrho = \sum p_i |\psi_i\rangle\langle\psi_i|$).

Conditions (first set) for a symmetry $\varrho \mapsto S(\varrho)$:

- $0 \leqslant \varrho \mapsto S(\varrho) \geqslant 0$,
- $\mathrm{Tr}_V\, S(\varrho_f)\, S(\varrho_i) = \mathrm{Tr}_V\, \varrho_f\, \varrho_i$.

Wigner's Theorem. S "lifts" to a transformation $\hat{S} : V \to V$, $S(\varrho) = \hat{S}\, \varrho\, \hat{S}^{-1}$,

which is either unitary or anti-unitary.

$$\langle \hat{S}\psi_f , \hat{S}\psi_i \rangle = \overline{\langle \psi_f , \psi_i \rangle} .$$

What's a symmetry in quantum mechanics? (II)

— associative algebra of observables $\mathcal{A}$.

— pairing : observables $\otimes$ states $\longrightarrow \mathbb{R}$,

$$A \otimes \rho \longmapsto \mathrm{Tr}_V\, \rho A .$$

— invariance of pairing : $\mathrm{Tr}_V\, \rho A = \mathrm{Tr}_V\, S(\rho)\, S'(A)$

$$\Longrightarrow \quad S'(A) = \hat{S} A \hat{S}^{-1}.$$

— Hamiltonian : $H = H^\dagger \geqslant 0$ (wlog).

Comment.

S' is an algebra automorphism.

Conditions (second set) on symmetries :

- $H = S'(H)$, or $H\hat{S} = \hat{S} H$.
- dynamics : $\hat{S}$ maps solutions of the equations of motion to solutions.

Comment on anti-unitary symmetries (say $\psi \mapsto T\psi$):

$$\left.\begin{cases} T H T^{-1} = +H \\ T\, i\, T^{-1} = -i \end{cases} \text{and} \quad i\hbar \frac{\partial}{\partial t}\psi = H\psi \right\} \Longrightarrow (T\psi)(x,t) = \overline{\psi(x,-t)} .$$

Non-Symmetries

- Gauge invariance
  - can never be broken, neither explicitly nor spontaneously (whereas symmetries can be).
  - does not entail any conservation laws (whereas symmetries do).

Note: the law of electric charge conservation should * NOT * be attributed to $U(1)$ electromagnetic gauge invariance.

- Bogolinbov – de Gennes "symmetry" of the BdG / Gorkov / Nambu formalism: $\mathcal{H} = -\Sigma_1 \overline{\mathcal{H}} \Sigma_1$,
$$\Sigma_1 = 1 \otimes (\sigma_1)_{ph} .$$

- Sublattice (or chiral) "symmetry": $\mathcal{H} = -\Sigma_3 \mathcal{H} \Sigma_3$,
where $V = V_A \oplus V_B$, $\psi = \psi_A + \psi_B \xmapsto{\Sigma_3} \psi_A - \psi_B$.

Particle-hole symmetry from sublattice "symmetry"

For a first-quantized Hamiltonian odd w.r.t. $V = V_A \oplus V_B$ consider

$$H = \sum_{ij} \left(c^\dagger_{Ai} H_{Ai,Bj} c_{Bj} + c^\dagger_{Bj} H_{Bj,Ai} c_{Ai} \right).$$

Let $K(c_{Ai}) = + c^\dagger_{Ai}$, $K(c_{Bj}) = - c^\dagger_{Bj}$, $K^2 = \mathrm{Id}$, $K(H_{Ai,Bj}) = \overline{H_{Ai,Bj}}$.

Then K is an anti-unitary symmetry: $\boxed{K(H) = H}$.

Example. $H_{kin} = \sum_{n \in \mathbb{Z}} \left(t_{n+1,n} c^\dagger_{n+1} c_n + t_{n,n+1} c^\dagger_n c_{n+1} \right)$, $t_{n,n+1} = \overline{t_{n+1,n}}$.

$K(c_n) = (-1)^n c^\dagger_n$, $K(c^\dagger_n) = (-1)^n c_n$, $K(i) = -i$.

Then $K(H_{kin}) = H_{kin}$.

<u>Remark</u>. $Q_{exc} = \frac{1}{2} \sum_\lambda (c^\dagger_\lambda c_\lambda - c_\lambda c^\dagger_\lambda) \equiv \sum_\lambda :c^\dagger_\lambda c_\lambda: = Q - \langle Q \rangle$

(excess charge w.r.t. half-filling) is particle-hole odd: $K(Q_{exc}) = - Q_{exc}$.

Particle-hole transformation versus T, C

The anti-unitary operations of K and T complement each other:

$$\begin{array}{ccccc} & & (\rho, j; E, B) & & \\ & \swarrow^{K} & & \searrow^{T} & \\ (-\rho, j; -E, B) & & & & (\rho, -j; E, -B). \end{array}$$

Note / Warning. Particle-hole transformation K
$\neq$ charge conjugation C.

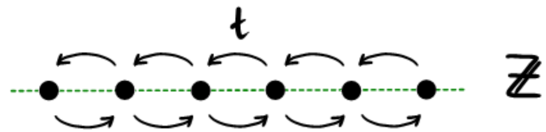
Indeed, C is a unitary transformation such that

$$(\rho, j; E, B) \xmapsto{\;C\;} (-\rho, -j; -E, -B).$$

(btw, C of little relevance to condensed matter physics).

Remark. There is some similarity between K and CT.

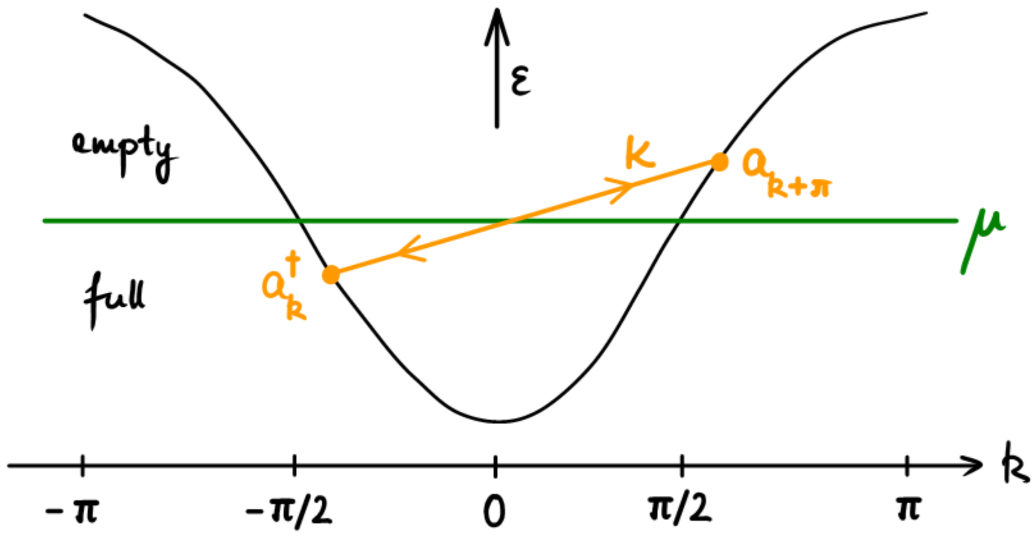
Example: cosine band



$$a_k = \sum_{n \in \mathbb{Z}} e^{-ikn} c_n \,, \quad c_n = \int_{-\pi}^{+\pi} \frac{dk}{2\pi} e^{ikn} a_k \,.$$

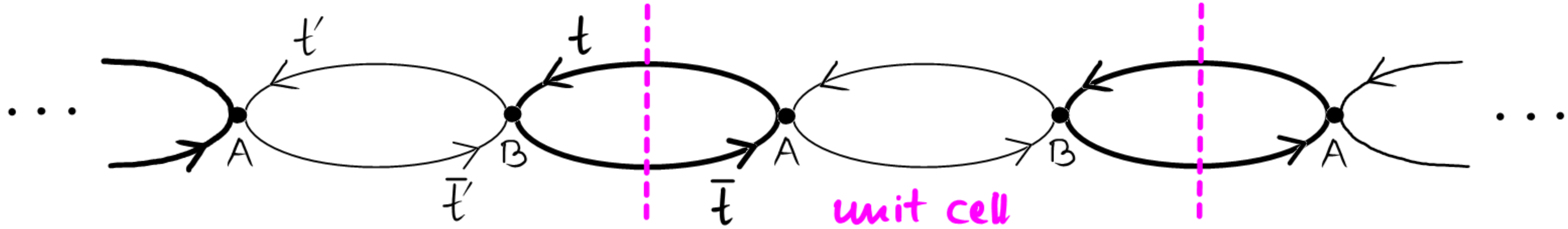
$$H - \mu N = \int \frac{dk}{2\pi} \varepsilon(k)\, a_k^\dagger a_k \,, \quad \varepsilon(k) = -\, t \cos k \,,$$

has the particle-hole symmetry $\quad a_k^\dagger \overset{K}{\longleftrightarrow} a_{k+\pi} \,.$



More generally, any band $\varepsilon(k+\pi) = -\,\varepsilon(k)$ with $\int \varepsilon(k)\, dk = 0$ has the same particle-hole symmetry.

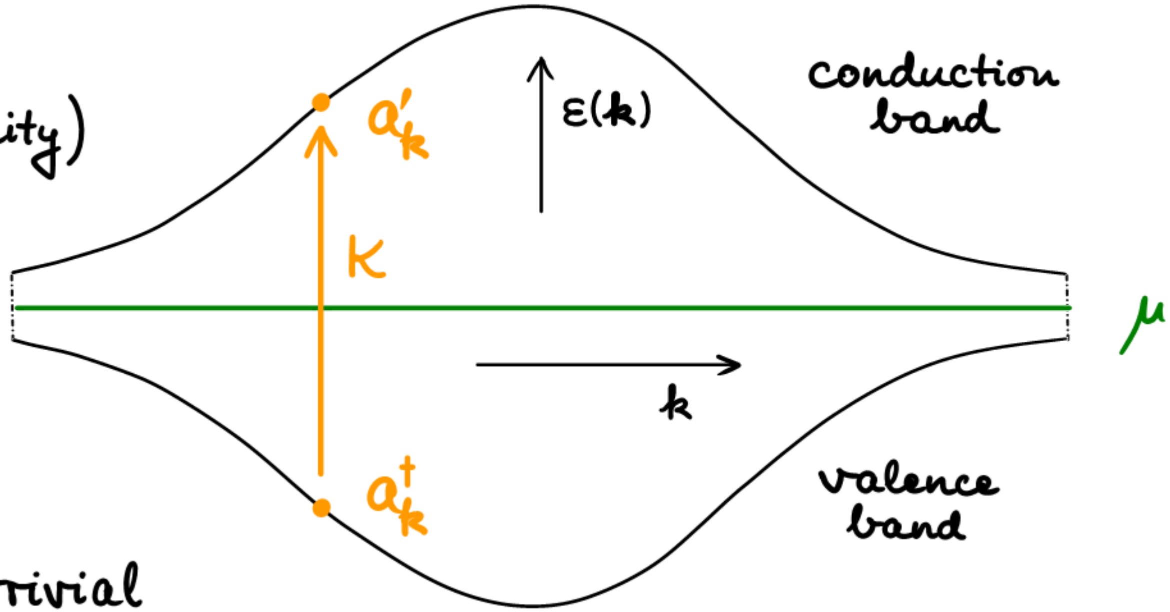
Su-Schrieffer-Heeger model ("polyacetylene"; AZ-class AIII)



$|t| > |t'|$ (Peierls instability)

doubled unit cell

↝ backfolding:



Valence band carries non-trivial topological invariant protected by particle-hole symmetry.

Bulk-boundary correspondence $\Longrightarrow$ gapless edge mode:

$$\psi = \sum_{n\geq 0}^{\infty} (-t'/t)^n \, c^\dagger_{2n}, \quad [H_{\geq 0}, \psi] = 0.$$

Formal structure: particle-hole conjugation

— Field operator $\psi = \sum (u_\lambda c_\lambda + v_\lambda c_\lambda^\dagger)$ $\quad (u_\lambda, v_\lambda \in \mathbb{C})$.

— Hermitian conjugation: $\psi \mapsto \psi^\dagger$, $(AB)^\dagger = B^\dagger A^\dagger$;

— particle-hole conjugation: $\psi \mapsto \psi^\dagger \equiv \psi^\flat$, $\underbrace{(AB)^\flat = A^\flat B^\flat}_{\text{automorphism}}$.

<u>Note</u>: every self-adjoint one-body Hamiltonian H (Weyl-ordered) is ph-odd: $H = \frac{1}{2}\sum_{\lambda\lambda'} \left(h_{\lambda\lambda'} (c_\lambda^\dagger c_{\lambda'} - c_{\lambda'} c_\lambda^\dagger) + \Delta_{\lambda\lambda'} c_\lambda^\dagger c_{\lambda'}^\dagger + \overline{\Delta_{\lambda\lambda'}} c_{\lambda'} c_\lambda \right) = -H^\flat$.

! Tautology! (BdG-"symmetry" is <u>not</u> a particle-hole symmetry.)

In contrast, particle-hole symmetry K is a concatenation of two operations,

$$K : H \xrightarrow{\flat} -H \xrightarrow{\Gamma} +H$$

(e.g. sublattice "symmetry" Γ or momentum shift $k \xrightarrow{\Gamma} k+\pi$).

Lifting particle-hole conjugation to Fock space

We seek Ξ such that $A^\flat = \Xi A \Xi^{-1}$.

Naive picture of Ξ in occupation number representation:

$$c^\dagger_{i_1} c^\dagger_{i_2} \cdots c^\dagger_{i_n} |\mathrm{vac}\rangle \xrightarrow{\ \Xi\ } \pm\, c_{i_1} c_{i_2} \cdots c_{i_n} |\mathrm{sea}\rangle$$

Pictorially: $\Xi \downarrow$

● ● ○ ● ○ ○

○ ○ ● ○ ● ●

¿ SIGN ?

Basis-independent formulation $\left(\mathcal{F}_n \equiv \Lambda^n(V),\ N = \dim V \right)$.

$$\Xi_n : \mathcal{F}_n \xrightarrow{\ \gamma_n\ } \mathcal{F}_n^* \xrightarrow{\ \omega_n\ } \mathcal{F}_{N-n}$$

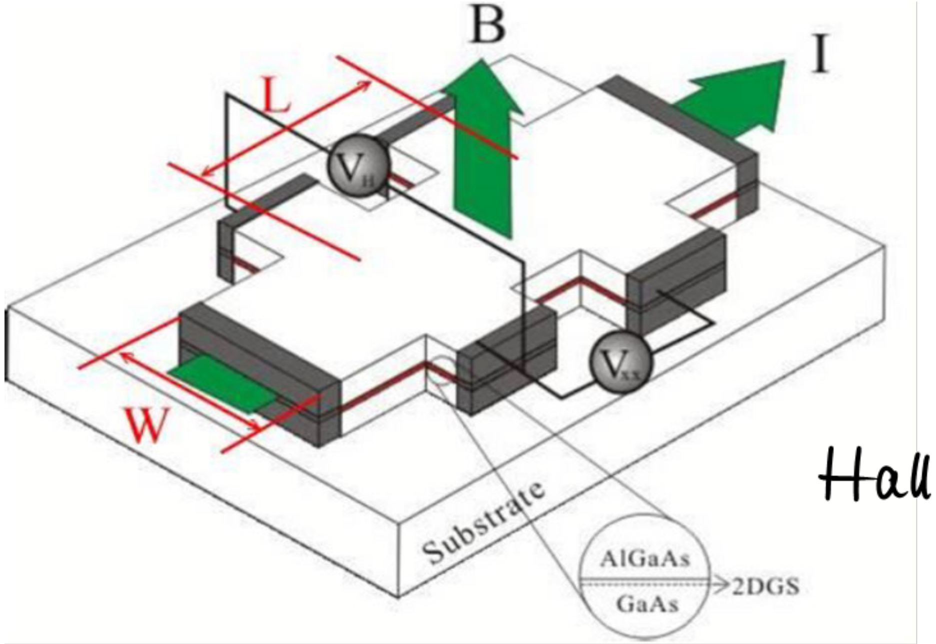
where γ_n Fréchet–Riesz isomorphism (a.k.a. Dirac ket-to-bra bijection)

and $\omega_n : \mathcal{F}_n^* \longrightarrow \mathcal{F}_{N-n}$ "wedge" isomorphism by $\psi_n \wedge \psi_{N-n} = B(\psi_n, \psi_{N-n})\, \psi_{\mathrm{sea}}$.

<u>Note</u>: Ξ cannot ever be symmetry of a Fermi-liquid ground state.

Half-filled
Lowest Landau Level

Quantum Hall Effect



Two-dimensional electron gas at low temperature and in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \dfrac{h}{\nu e^2}$.

$N_\phi = \frac{e}{h} \iint B$ = # of states in lowest Landau level (LLL).

$\nu = N_{el}/N_\phi$ filling fraction; $\nu = 1/2$ half-filled LLL.

Particle-hole conjugation symmetry: $\sigma_{xx}(\nu) = \sigma_{xx}(1-\nu)$ is exact in the limit $\hbar\omega_c \longrightarrow \infty$ and if interactions are two-body.

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conjugation

Particle-hole symmetry in the anomalous quantum Hall effect

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(Received 27 February 1984)

This paper explores the uses of particle-hole symmetry in the study of the anomalous quantum Hall effect. A rigorous algorithm is presented for generating the particle-hole dual of any state. This is used to derive Laughlin's quasihole state from first principles and to show that this state is exact in the limit $\nu \to 1$, where ν is the Landau-level filling factor. It is also rigorously demonstrated that the creation of m quasiholes in Laughlin's state with $\nu = 1/m$ is precisely equivalent to creation of one true hole. The charge-conjugation procedure is also generalized to obtain an algorithm for the generation of a hierarchy of states of arbitrary rational filling factors.

I. INTRODUCTION

The anomalous quantum Hall effect[1,2] is one of the most striking many-body phenomena discovered in recent years. The Hall resistivity of a two-dimensional electron gas (inversion layer) in a high magnetic field at low temperatures exhibits quantized plateau values of the form $\rho_{xy} = h/e^2 i$, where i is a rational number $i = p/q$ with q odd. Associated with this quantization of the Hall resistivity is a marked decrease in the dissipation $(\rho_{xx} \to 0)$. The latter suggests the

where the exponential factors have been lumped into the measure

$$d\mu(z) = \frac{dx\,dy}{2\pi l^2} e^{-|z|^2/2} \quad . \tag{4}$$

Within (the N-particle version of) this space the variational wave functions proposed by Laughlin[3] may be written

$$\psi_m(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m \quad , \tag{5}$$

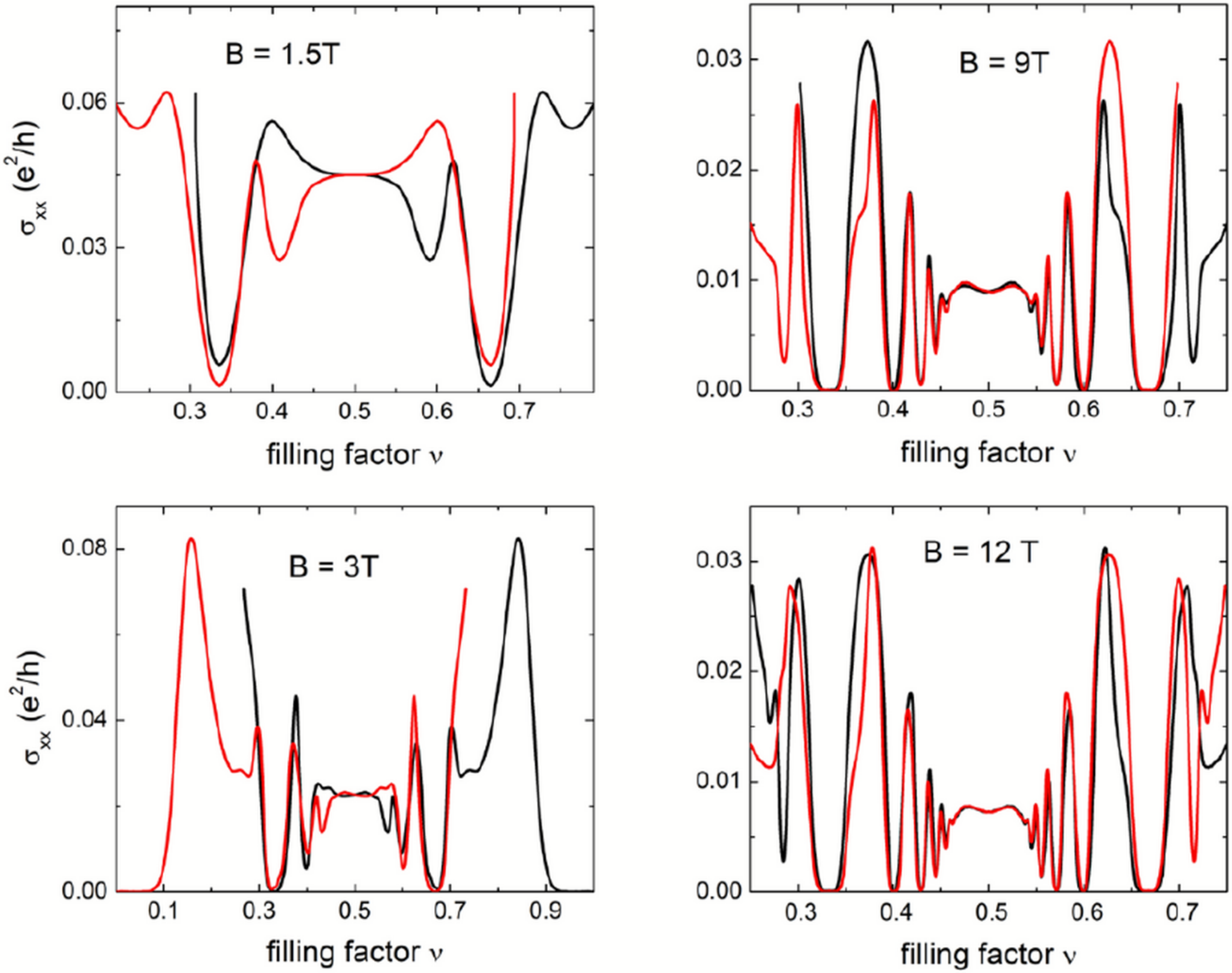

Figure 4. Examination of reflection symmetry in σ_{xx} over a large range of magnetic fields, from 1.5T to 12T.

Pan, Kang, Lilly, Reno, Baldwin, West, Pfeiffer, Tsui (March 2019)

Theory of the half-filled Landau level

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A two-dimensional electron system in an external magnetic field, with Landau-level filling factor $\nu=\frac{1}{2}$, can be transformed to a mathematically equivalent system of fermions interacting with a Chern-Simons gauge field such that the average effective magnetic field acting on the fermions is zero. If one ignores fluctuations in the gauge field, this implies that for a system with no impurity scattering, there should be a *well-defined Fermi surface* for the fermions. When gauge fluctuations are taken into account, we find that there can be infrared divergent corrections to the quasiparticle propagator, which we interpret as a divergence in the effective mass m^*, whose form depends on the nature of the assumed electron-electron interaction $v(\mathbf{r})$. For long-range interactions that fall off slower than $1/r$ at large separation r, we find no infrared divergences; for short-range repulsive interactions, we find power-law divergences; while for Coulomb interactions, we find logarithmic corrections to m^*. Nevertheless, we argue that many features of the Fermi surface are likely to exist in all these cases. In the presence of a weak impurity-scattering potential, we predict a finite resistivity ρ_{xx} at low temperatures, whose value we can estimate. We compute an anomaly in surface acoustic wave propagation that agrees qualitatively with recent experiments. We also make predictions for the size of the energy gap in the fractional quantized Hall state at $\nu=p/(2p+1)$, where p is an integer. Finally, we discuss the implications of our picture for the electronic specific heat and various other physical properties at $\nu=\frac{1}{2}$, we discuss the generalization to other filling fractions with even denominators, and we discuss the overall phase diagram that results from combining our picture with previous theories that apply to the regime where impurity scattering is dominant.

Halperin, Lee, Read (1993)

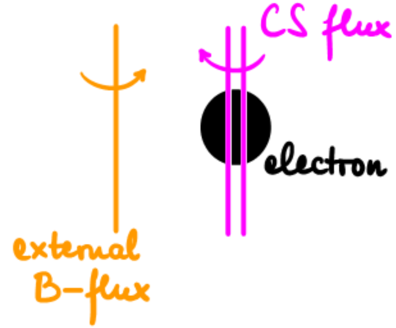
Composite fermions (i.e. electrons with 2 fictitious magnetic flux quanta attached) experience zero net B-field and populate a Fermi sea (in free-fermion approximation).

⇝ Effective field theory with Lagrangian:

$$\mathcal{L} = i\psi^\dagger(\partial_t - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \ldots$$

Chern Simons term

Phenomenologically quite successful!



Problems: – mass renormalization?

– particle-hole symmetry?

<u>Note</u>. $H_{\text{eff}} = P_{LLL} V_2 P_{LLL}$, V_2 quadratic in charge density, is exactly particle-hole conjugation symmetric: $H_{\text{eff}} = +\Xi H_{\text{eff}} \Xi^{-1}$.

Remark. Son's new theory resolves both of these issues!

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Is the Composite Fermion a Dirac Particle?

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We propose a particle-hole symmetric theory of the Fermi-liquid ground state of a half-filled Landau level. This theory should be applicable for a Dirac fermion in the magnetic field at charge neutrality, as well as for the $\nu = \frac{1}{2}$ quantum Hall ground state of nonrelativistic fermions in the limit of negligible inter-Landau-level mixing. We argue that when particle-hole symmetry is exact, the composite fermion is a massless Dirac fermion, characterized by a Berry phase of π around the Fermi circle. We write down a tentative effective field theory of such a fermion and discuss the discrete symmetries, in particular, $\mathcal{CP}$. The Dirac composite fermions interact through a gauge, but non-Chern-Simons, interaction. The particle-hole conjugate pair of Jain-sequence states at filling factors $n/(2n+1)$ and $(n+1)/(2n+1)$, which in the conventional composite fermion picture corresponds to integer quantum Hall states with different filling factors, n and $n+1$, is now mapped to the same half-integer filling factor $n + \frac{1}{2}$ of the Dirac composite fermion. The Pfaffian and anti-Pfaffian states are interpreted as d-wave Bardeen-Cooper-Schrieffer paired states of the Dirac fermion with orbital angular momentum of opposite signs, while s-wave pairing would give rise to a particle-hole symmetric non-Abelian gapped phase. When particle-hole symmetry is not exact, the Dirac fermion has a $\mathcal{CP}$-breaking mass. The conventional fermionic Chern-Simons theory is shown to emerge in the nonrelativistic limit of the massive theory.

DOI: 10.1103/PhysRevX.5.031027 Subject Areas: Condensed Matter Physics

Question: $H_{eff} \overset{\checkmark}{=} + \Xi H_{eff} \Xi^{-1}$, but $\Xi |FL\rangle \overset{?}{=} |FL\rangle$.

Son's Logic

- Realize lowest Landau level as zero-energy sector of Dirac fermion (in a homogeneous magnetic field): $S = \int d^3x i\bar{\Psi}\gamma^\mu(\partial_\mu - iA_\mu)\Psi$ + $S_{E.M.}$

  $\curvearrowright$ ph-conjugation Ξ is implemented as CT (charge conjugation & time reversal)

- Switch to dual description (QED$_3$; fermionic particle-vortex duality) by $\bar{\Psi}\gamma^\mu\Psi = J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

  (a_0 = magnetization, $\epsilon^{ij}a_j$ = electric polarization field).

  $$S_{\text{eff}} = \int d^3x \left(i\bar{\psi}\gamma^\mu(\partial_\mu + 2ia_\mu)\psi + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \right) + S_{E.M.}.$$

  Note. No mass term, no Chern-Simons term!

Key observation. In the dual formulation, $\Xi \hat{=} CT$ acts like T, thereby making a Fermi-liquid ground state $|FL\rangle = T|FL\rangle$ possible.

Symmetry considerations

- Electromagnetic gauge field $A = A_\mu dx^\mu$ is time-twisted:
  under time reversal $A \longmapsto -T^*A$ or $A_0 \longmapsto +A_0$, $A_j \longmapsto -A_j$
  and under parity $A \longmapsto +P^*A$ or $A_0 \longmapsto +A_0$, ...
- Charge 3-current $J = da$ is space-twisted. Hence
  time reversal : $a \longmapsto +T^*a$
  parity : $a \longmapsto -P^*a$
  $\rightsquigarrow \int A \wedge J$ invariant

Consequence: First-quantized Hamiltonian for the fermionic vortex field

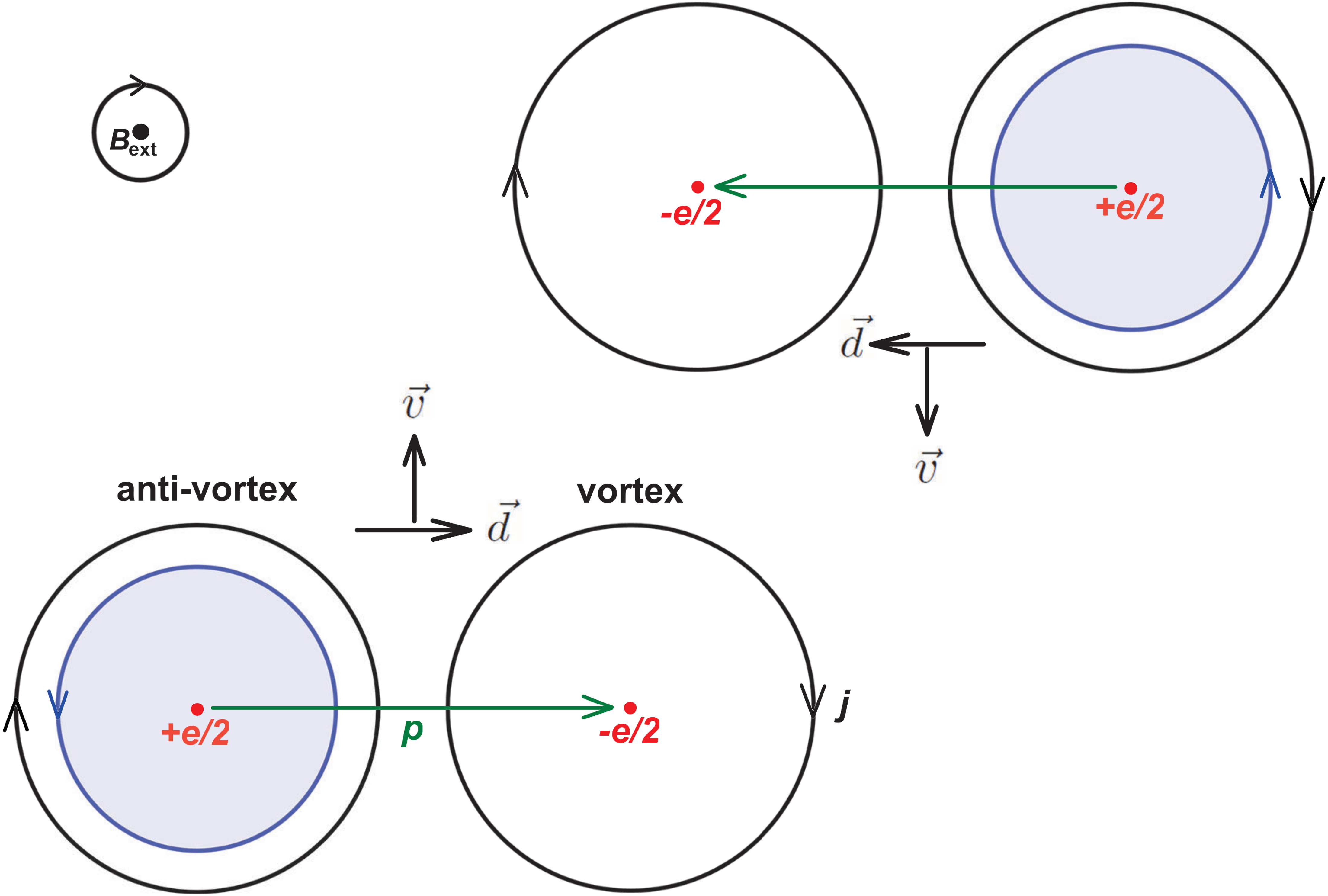
$$H = v\sigma_1(p_1 + 2a_1) + v\sigma_2(p_2 + 2a_2) + a_0$$

is <u>odd</u> under each of time reversal, parity, and charge conjugation.

<u>Time reversal</u>: $\psi \overset{\mathbb{C}\text{-linear}}{\longmapsto} \sigma_3\psi$, $a_0 \mapsto -a_0$, $a_1 \mapsto +a_1$, $a_2 \mapsto +a_2$.

<u>Parity</u> $(x_2 \mapsto -x_2)$: $\psi \overset{\mathbb{C}\text{-antilinear}}{\longmapsto} \bar{\psi}$, $a_0 \mapsto -a_0$, $a_1 \mapsto -a_1$, $a_2 \mapsto +a_2$.

<u>Charge conjugation</u>: $\psi \overset{\mathbb{C}\text{-antilinear}}{\longmapsto} \sigma_1\bar{\psi}$, $a_\mu \mapsto -a_\mu$.



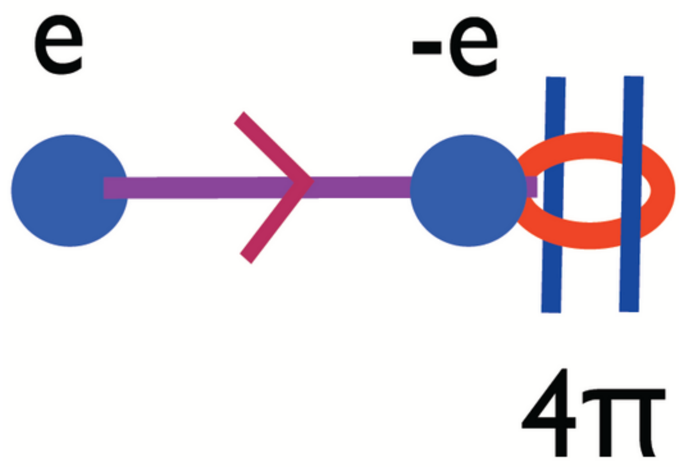
B_{ext}
-e/2
+e/2
$\vec{d}$
$\vec{v}$
anti-vortex
vortex
+e/2
p
-e/2
j

Homework Questions

- What is the statement of Wigner's Theorem?
- If H fails to be positive, then H is not the Hamiltonian. Why not?
- How does charge conjugation differ from a particle-hole transformation?
- Why don't we speak of gauge invariance as a symmetry?
- Does every superconductor have particle-hole symmetry?
- What is the Su-Schrieffer-Heeger model? Describe its particle-hole symmetry!
- In what sense is particle-hole symmetry complementary to time-reversal symmetry?
- What's the difference between particle-hole conjugation and a ph-transformation?
- How does particle-hole conjugation differ from Hermitian conjugation?
- Why are Fermi-liquid states never particle-hole conjugation symmetric? How does Son's proposal for a Fermi-liquid ground state of the half-filled lowest Landau level get around this obstruction?
- What are the main differences between Son's proposal and the traditional theory of Halperin-Lee-Read?
- What do we mean when we say that the gauge field A is "time-twisted"?

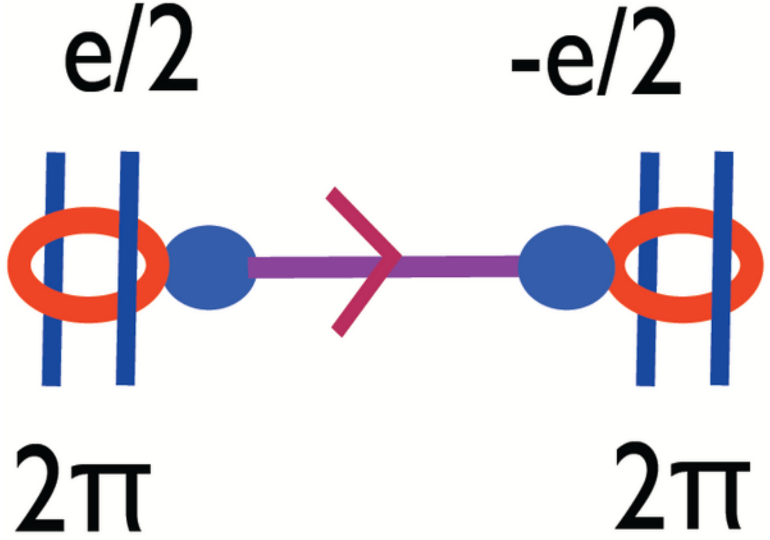
Pictures of composite fermion as a dipole

Wang & Senthil (2016)



traditional picture due to Halperin-Lee-Read

new picture of CF with particle-hole symmetry



Fermionic particle–vortex duality

! exchanges particle–hole and time–reversal symmetry !

- $S = \int dt \int d^2x \; \bar{\psi} \, \gamma^\mu (\partial_\mu + A_\mu) \psi \; + \; S_{E.M.}[A]$ is dual to

$$S_{dual} = \int dt \int d^2x \; \bar{\xi} \, \gamma^\mu (\partial_\mu + a_\mu) \xi \; + \int A \wedge da \; + \; S_{E.M.}[A] \, .$$

- Particle–hole symmetry of S,

$$K : \; \psi(x,t) \longmapsto \sigma_3 \, \psi^\dagger(x,-t) \, , \qquad \begin{aligned} A_0(x,t) &\longmapsto - A_0(x,-t) \, , \\ A_i(x,t) &\longmapsto + A_i(x,-t) \, , \end{aligned}$$

turns into time–reversal symmetry for S_{dual} ,

$$\tilde{K} : \; \xi(x,t) \longmapsto i\sigma_2 \, \xi(x,-t) \, , \qquad \begin{aligned} a_0(x,t) &\longmapsto + a_0(x,-t) \, , \\ a_i(x,t) &\longmapsto - a_i(x,-t) \, . \end{aligned}$$

(Note: $da \equiv J$ = 2– form of the charge current)

Application (D.T. Son, PRX 2016) to half–filled lowest Landau level.