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DRESDEN concept

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Ginzburg-Landau Theory of Superconductivity

- Introduction
- Landau theory of phase transitions
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- Ginzburg-Landau theory for superconductors

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Introduction

Theories of superconductivity

- London and London (1935): macroscopic electrodynamics for superconductors (augments Maxwell equations by constitutive equations), describes
  - ideal conductivity
  - flux expulsion (Meißner-Ochsenfeld effect)
- Ginzburg and Landau (1950): phenomenological theory based on Landau's theory of phase transitions, advantage: independent of microscopic mechanism
- Bardeen, Cooper, and Schrieffer (1957): microscopic mean-field (BCS) theory
- Gor'kov (1959): derivation of Ginzburg-Landau theory from BCS theory

Landau theory of phase transitions

- define order parameter Δ: $\Delta = 0$ on one side of the phase transition, $\Delta \neq 0$ on the other
- write appropriate thermodynamic potential ("free energy") as function of Δ and other thermodynamic variables (temperature, $\mathbf{B}$-field, ...): $F = F(\Delta, T, \mathbf{B}, \dots)$
- mean-field approximation: equilibrium state is the ensemble state that makes F stationary: $\partial F / \partial \Delta = 0$ (neglect fluctuations)
- Landau's idea: expand $F(\Delta, T, \mathbf{B}, \dots)$ in powers of $\Delta, \mathbf{B}, \dots$
  - include only terms allowed by symmetry
  - keep minimum number of terms necessary to describe the relevant physics

Example: ferromagnetic Heisenberg model

- order parameter: magnetization $\mathbf{M} \in \mathbb{R}^3$
- system invariant under rotations of $\mathbf{M}$: $\mathrm{SO}(3)$ symmetry
- system invariant under time reversal $\mathbf{M} \mapsto -\mathbf{M}$
- expand smooth function F in power series about $\mathbf{M} = 0$:

$$F = F_0 + \alpha\, \mathbf{M} \cdot \mathbf{M} + \frac{\beta}{2}\, (\mathbf{M} \cdot \mathbf{M})^2 + \cdots$$

  $\beta > 0$ to make bounded from below, F_0, α, β are functions of T etc.
- F_0 is free energy of nonmagnetic state $\longrightarrow$ irrelevant for phase transition $\longrightarrow$ drop;

$$F = \alpha\, \mathbf{M} \cdot \mathbf{M} + \frac{\beta}{2}\, (\mathbf{M} \cdot \mathbf{M})^2 + \cdots$$

  is the free-energy *difference* between magnetic and nonmagnetic states

What does this look like for a superfluid?

- Ginzburg and Landau: in analogy with Bose-Einstein condensate, superfluid is described by a single-particle wave function $\Psi_s(\mathbf{r})$!
- choose amplitude $\psi \in \mathbb{C}$ of $\Psi_s(\mathbf{r})$ as order parameter (neglect variation of $\Psi_s(\mathbf{r})$ on atomic scales)
- F does not depend on phase of ψ since global phase of $\Psi_s(\mathbf{r})$ not observable: $\mathrm{U}(1)$ symmetry
- expansion (dropping higher orders):

$$F = \alpha\, \psi^* \psi + \frac{\beta}{2}\, (\psi^* \psi)^2 = \alpha\, |\psi|^2 + \frac{\beta}{2}\, |\psi|^4$$

$$F = \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 \qquad \text{with } \beta > 0$$

Two cases:

- $\alpha \geq 0$: F has single minimum at $\psi = 0$, normal state
- $\alpha < 0$: F has ring of minima

$$\frac{\partial F}{\partial |\psi|} = 2\alpha\,|\psi| + 2\beta\,|\psi|^3 = 0$$

thus

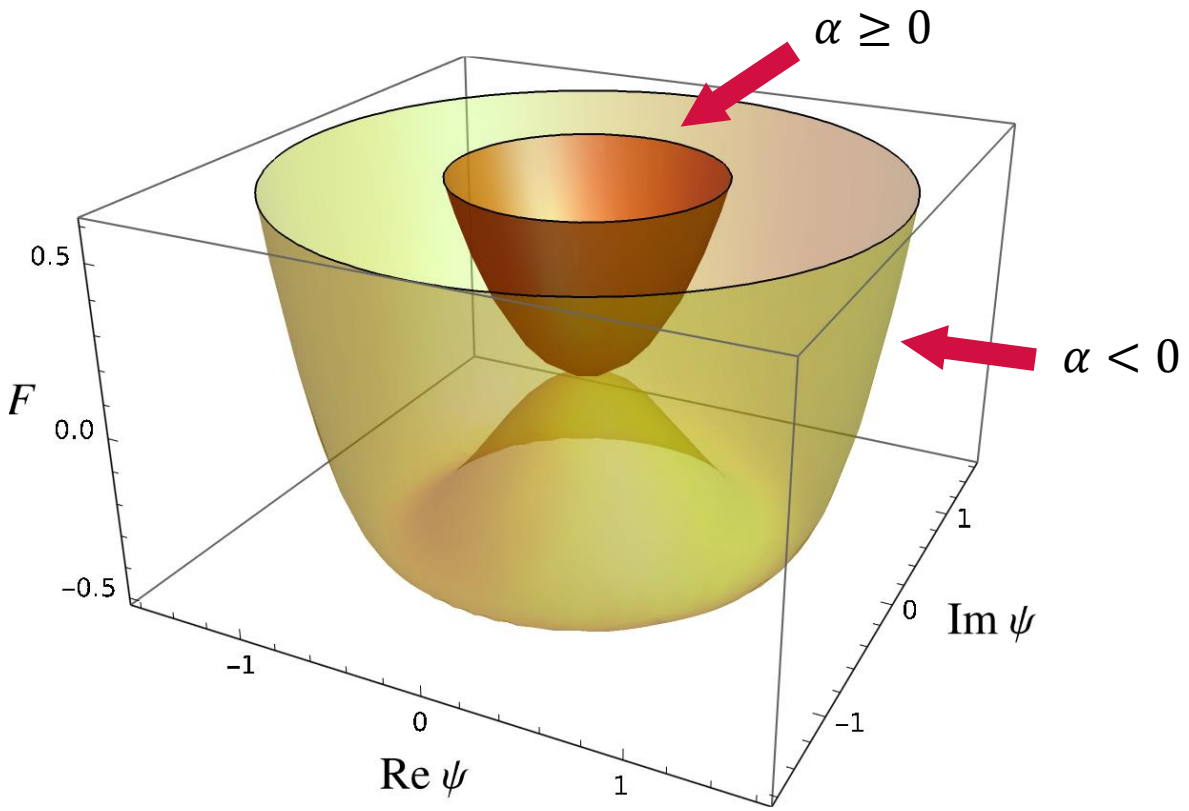
$$|\psi| = \sqrt{-\alpha/\beta}$$

Phase transition at $\alpha = 0$, expand in T:

$$\alpha \cong \alpha'\,(T - T_c)$$
$$\beta \cong \text{const}$$

then

$$|\psi| \cong \sqrt{\alpha'/\beta}\,\sqrt{T_c - T} \quad \text{for } T \leq T_c$$



"Mexican-hat" or "wine-bottle" potential

exponent ½ typical for mean-field theories

$$F = \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 \qquad \text{with } \beta > 0$$

- true thermodynamic free energy equals F evaluated at the equilibrium solution for ψ
- obtain further thermodynamic variables as derivatives, e.g., superfluid contribution to entropy for $T < T_c$:

$$S_s = -\frac{\partial}{\partial T}\,F\left(\sqrt{-\alpha/\beta}\right) \cong -\frac{(\alpha')^2}{\beta}\,(T_c - T) < 0,$$

contribution to heat capacity:

$$C_s = T\,\frac{\partial S_s}{\partial T} = \begin{cases} \dfrac{(\alpha')^2}{\beta}\,T & \text{for } T < T_c \\ 0 & \text{for } T \geq T_c \end{cases}$$

jump discontinuity at T_c

- … in second derivative $\partial^2 F/\partial T^2 \rightarrow$ second-order phase transition

Ginzburg-Landau theory for neutral superfluids

- Ginzburg and Landau: allow for spatially nonuniform situations!
- include gradient terms in the expansion of the free energy
- define "macroscopic condensate wave function" $\psi(\mathbf{r})$ as $\Psi_s(\mathbf{r})$ averaged over length scales large compared to atomic distances
- expansion:

$$F[\psi] = \int d^3r \left[\alpha\,|\psi(\mathbf{r})|^2 + \frac{\beta}{2}\,|\psi(\mathbf{r})|^4 + \gamma\left(\nabla\psi(\mathbf{r})\right)^* \cdot \nabla\psi(\mathbf{r})\right]$$

  (a *functional* of $\psi(\mathbf{r})$)

- expect $\gamma > 0$ since modulations should cost energy
- identify gradient term as "kinetic energy": $\gamma = \hbar^2/2m^*$ ⟵ effective mass m^*
- equilibrium: ensemble state that makes F stationary (mean-field approximation)

quite good for superconductors

- find the mean-field solution:

$$\frac{\delta F}{\delta \psi^*(\mathbf{r})} = 0$$

(a *functional derivative*; compare derivation of Lagrange equation from Hamilton's principle)

- integration by parts:

$$\frac{\delta F[\psi(\mathbf{r}')]}{\delta \psi^*(\mathbf{r})} = \frac{\delta}{\delta \psi^*(\mathbf{r})} \int d^3r' \left[\alpha\, \psi^*(\mathbf{r}')\psi(\mathbf{r}') + \frac{\beta}{2}\, \psi^*(\mathbf{r}')^2 \psi(\mathbf{r}')^2 - \frac{\hbar^2}{2m^*}\, \psi^*(\mathbf{r}')(\nabla')^2 \psi(\mathbf{r}') \right]$$

$$= \int d^3r' \left[\alpha\, \delta(\mathbf{r}' - \mathbf{r})\psi(\mathbf{r}') + \beta\, \delta(\mathbf{r}' - \mathbf{r})\psi^*(\mathbf{r}')\psi(\mathbf{r}')^2 - \frac{\hbar^2}{2m^*}\, \delta(\mathbf{r}' - \mathbf{r})(\nabla')^2 \psi(\mathbf{r}') \right]$$

$$= \alpha\, \psi(\mathbf{r}) + \beta\, \psi^*(\mathbf{r})\psi^2(\mathbf{r}) - \frac{\hbar^2}{2m^*}\, \nabla^2 \psi(\mathbf{r}) = 0$$

thus

$$-\frac{\hbar^2}{2m^*}\, \nabla^2 \psi + \alpha\, \psi + \beta\, |\psi|^2 \psi = 0$$

Example: superfluid filling half space $x > 0$

$$-\frac{\hbar^2}{2m^*}\,\psi''(x) + \alpha\,\psi(x) + \beta\,|\psi(x)|^2\,\psi(x) = 0$$

- boundary conditions:

$$\psi(x = 0) = 0$$
$$\lim_{x\to\infty} \psi(x) = \sqrt{-\alpha/\beta} \qquad \text{(uniform equilibrium value)}$$

- let $\psi(x) = \sqrt{-\alpha/\beta}\; f(x)$, then

$$\underbrace{-\frac{\hbar^2}{2m^*\alpha}}_{\equiv \xi^2 > 0} f''(x) + f(x) - f(x)^3 = 0$$

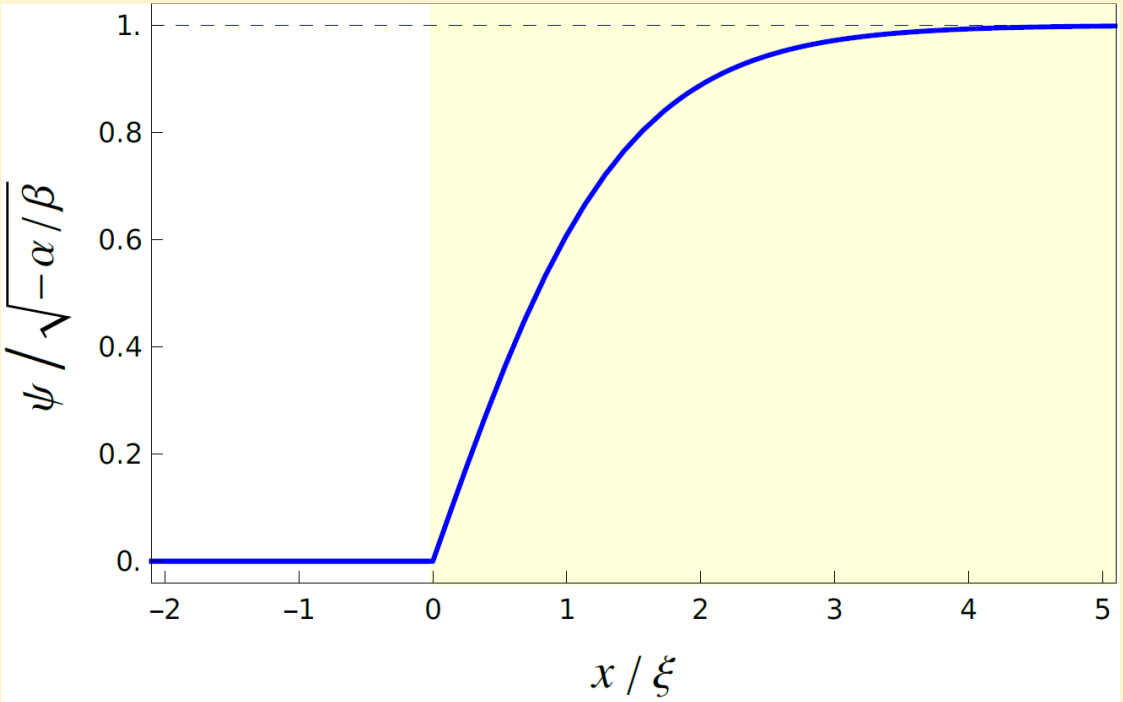
- contains length scale: Ginzburg-Landau coherence length

$$\xi = \sqrt{-\frac{\hbar^2}{2m^*\alpha}} \cong \sqrt{\frac{\hbar^2}{2m^*\alpha'\,(T_c - T)}}$$

- nonlinear differential equation but does have an exact solution:

$$f(x) = \psi(x)/\sqrt{-\alpha/\beta} = \tanh\frac{x}{\sqrt{2}\,\xi}$$

for $x \geq 0$



Ginzburg-Landau theory for neutral superfluids

Fluctuations for $T > T_c$

- so far only equilibrium state ψ_0
- at finite temperatures, ψ fluctuates about ψ_0, still static ansatz:

$$\psi(\mathbf{r}) = \psi_0 + \delta\psi(\mathbf{r})$$

- for $T > T_c$: $\psi_0 = 0$ so that $\psi(\mathbf{r}) = \delta\psi(\mathbf{r})$
- treat F as *energy* for *macroscopic* modulations of $\psi(\mathbf{r})$ and write partition function:

$$\mathcal{Z} = \int D^2\psi \, e^{-F[\psi]/k_B T}$$

(a *functional integral* since continuum of modes, see Altland and Simons in lecture notes)

- hard to evaluate → Gaussian approximation: only keep F to second order in $\psi = \delta\psi$,

$$F[\psi] \cong \int d^3r \left[\alpha \, \psi^*(\mathbf{r})\psi(\mathbf{r}) + \frac{\hbar^2}{2m^*} \left(\frac{1}{i} \nabla\psi(\mathbf{r}) \right)^* \cdot \frac{1}{i} \nabla\psi(\mathbf{r}) \right]$$

- insert Fourier transform

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\, \psi_{\mathbf{k}}$$

obtain

$$F[\psi] \cong \frac{1}{V} \sum_{\mathbf{k},\mathbf{k}'} \underbrace{\int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}}}_{= V\delta_{\mathbf{k}\mathbf{k}'}} \left[\alpha\, \psi_{\mathbf{k}}^* \psi_{\mathbf{k}'} + \frac{\hbar^2}{2m^*} (\mathbf{k}\, \psi_{\mathbf{k}})^* \cdot \mathbf{k}' \psi_{\mathbf{k}'} \right] = \sum_{\mathbf{k}} \left(\alpha + \frac{\hbar^2 k^2}{2m^*} \right) \psi_{\mathbf{k}}^* \psi_{\mathbf{k}}$$

thus

$$\mathcal{Z} \cong \prod_{\mathbf{k}} \int_{\mathbb{C}} d^2\psi \exp\left(-\frac{1}{k_B T} \left(\alpha + \frac{\hbar^2 k^2}{2m^*} \right) \psi^* \psi \right) = \prod_{\mathbf{k}} \frac{\pi k_B T}{\alpha + \frac{\hbar^2 k^2}{2m^*}}$$

- can obtain thermodynamic quantities, e.g., heat capacity:

$$C_s = -T\, \frac{\partial^2 \mathcal{F}}{\partial T^2} = T\, \frac{\partial^2}{\partial T^2}\, k_B T \ln \mathcal{Z} \cong k_B T \frac{\partial^2}{\partial T^2}\, T \sum_{\mathbf{k}} \ln \frac{\pi k_B T}{\alpha + \frac{\hbar^2 k^2}{2m^*}}$$

consider only term singular at T_c, stems from $\alpha(T)$:

$$C_{\mathrm{crit}} \cong -k_B T^2 \frac{\partial^2}{\partial T^2} \sum_{\mathbf{k}} \ln\left(\alpha + \frac{\hbar^2 k^2}{2m^*}\right) \cong k_B T^2 \sum_{\mathbf{k}} \frac{(\alpha')^2}{\left(\alpha + \frac{\hbar^2 k^2}{2m^*}\right)^2}$$

in the thermodynamic limit, $V \to \infty$, go over to integral over $\mathbf{k}$,

$$C_{\mathrm{crit}} \cong \frac{k_B T^2}{2\sqrt{2}\,\pi} \frac{(m^*)^{3/2}}{\hbar^3} (\alpha')^2 V \frac{1}{\sqrt{\alpha}} \propto (T - T_c)^{-1/2}$$

exponent $-1/2$ typical for Gaussian approximation

(recall: at mean-field level, we found a step)

- for any $\mathbf{k}$, we find two fluctuation modes ($\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$) with dispersion

$$\epsilon_{\mathbf{k}} = \alpha + \frac{\hbar^2}{2m^*} k^2,$$

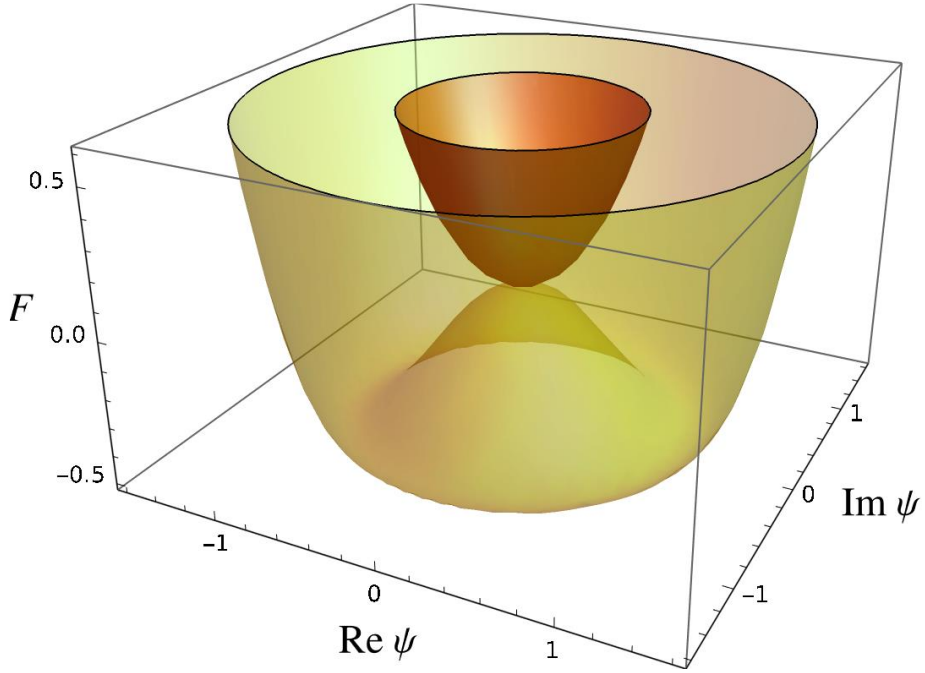
energy gap (mass) $\alpha > 0$!

Ginzburg-Landau theory for neutral superfluids

Fluctuations for $T < T_c$

- all states with $|\psi_0| = \sqrt{-\alpha/\beta}$ are equally good mean-field solutions
- phase fluctuations expected to have low energy since global change does not change F
- write

$$\psi(\mathbf{r}) = [\psi_0 + \delta\psi(\mathbf{r})]\, e^{i\phi(\mathbf{r})}, \qquad \psi_0 = \sqrt{-\alpha/\beta}$$



- Gaussian approximation:

$$F[\delta\psi,\phi] \cong \int d^3r \left[-2\alpha\,\delta\psi^2 + \frac{\hbar^2}{2m^*}\left(\frac{1}{i}\nabla\delta\psi\right)^* \cdot \frac{1}{i}\nabla\delta\psi - \frac{\hbar^2}{2m^*}\,\frac{\alpha}{\beta}\,(\nabla\phi)^* \cdot \nabla\phi\right]$$

$$= \sum_{\mathbf{k}} \left[\left(-2\alpha + \frac{\hbar^2 k^2}{2m^*}\right)\delta\psi_{\mathbf{k}}^*\delta\psi_{\mathbf{k}} - \frac{\alpha}{\beta}\,\frac{\hbar^2 k^2}{2m^*}\,\phi_{\mathbf{k}}^*\phi_{\mathbf{k}}\right]$$

amplitude modes nondegenerate and massive with gap $-2\alpha > 0$, phase fluctuations massless (Goldstone modes from broken $\mathrm{U}(1)$ symmetry)

- Remark: also for $T < T_c$ have

$$C_{\mathrm{crit}} \propto (T - T_c)^{-1/2}$$

Ginzburg-Landau theory for neutral superfluids

Time-dependent Ginzburg-Landau theory

- extension to describe dynamics: coarse-grained wave function $\psi(\mathbf{r}, t)$ is time dependent
- consider the action

$$\mathcal{S} = \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f} dt \int d^3r\, \mathcal{L}$$

with the Lagrange-density functional

$$\mathcal{L}\left[\psi, \nabla\psi, \dot{\psi}\right] = i\hbar\, \psi^* \frac{\partial\psi}{\partial t} - \alpha\, |\psi|^2 - \frac{\beta}{2}\, |\psi|^4 - \frac{\hbar^2}{2m^*}\, |\nabla\psi|^2$$

- variational principle $\delta\mathcal{S} = 0$ gives equations of motion

$$i\hbar\, \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m^*}\, \nabla^2\psi + \alpha\, \psi + \beta\, |\psi|^2\psi$$

time-dependent Ginzburg-Landau (or Gross-Pitaevski) equation

a nonlinear generalized Schrödinger equation

Application: rederive superfluid fluctuations for $T < T_c$

- integration by parts in $\mathcal{S}$:

$$\mathcal{L} = \frac{i\hbar}{2}\,\psi^*\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2}\,\frac{\partial\psi^*}{\partial t}\,\psi - \alpha\,|\psi|^2 - \frac{\beta}{2}\,|\psi|^4 - \frac{\hbar^2}{2m^*}\,|\nabla\psi|^2$$

- with $\psi = (\psi_0 + \delta\psi)\,e^{i\phi}$:

$$\frac{i\hbar}{2}\,\psi^*\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2}\,\frac{\partial\psi^*}{\partial t}\,\psi = \frac{i\hbar}{2}\,(\psi_0+\delta\psi)\,\frac{\partial\delta\psi}{\partial t} + \frac{i\hbar}{2}\,(\psi_0+\delta\psi)^2\,i\,\frac{\partial\phi}{\partial t}$$
$$- \frac{i\hbar}{2}\,\frac{\partial\delta\psi}{\partial t}\,(\psi_0+\delta\psi) - \frac{i\hbar}{2}\,(\psi_0+\delta\psi)^2\,(-i)\,\frac{\partial\phi}{\partial t}$$
$$= -\hbar\,(\psi_0+\delta\psi)^2\,\frac{\partial\phi}{\partial t}$$

and thus, at Gaussian level (second order),

$$\mathcal{L} \cong -2\hbar\sqrt{-\alpha/\beta}\;\delta\psi\,\frac{\partial\phi}{\partial t} + 2\alpha\,\delta\psi^2 - \frac{\hbar^2}{2m^*}\left(\frac{1}{i}\nabla\delta\psi\right)^*\cdot\frac{1}{i}\nabla\delta\psi - \frac{\hbar^2}{2m^*}\,\frac{\alpha}{\beta}\,(\nabla\phi)^*\cdot\nabla\phi$$

- the canonically conjugate field to the phase ϕ describes the amplitude fluctuations:

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -2\hbar\sqrt{-\alpha/\beta}\;\delta\psi$$

- go from Lagrangian to Hamiltonian description (cf. classical mechanics): Legendre transformation,

$$\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L} = -2\alpha\,\delta\psi^2 + \frac{\hbar^2}{2m^*}\left(\frac{1}{i}\nabla\delta\psi\right)^* \cdot \frac{1}{i}\nabla\delta\psi - \frac{\hbar^2}{2m^*}\,\frac{\alpha}{\beta}\,(\nabla\phi)^* \cdot \nabla\phi,$$

  as we know already

Ginzburg-Landau theory for superconductors

- particles forming the condensate carry charge $q \neq 0 \rightarrow$ couple to electromagnetic field:
  - replace gradient by gauge-covariant derivative (minimal coupling):

$$\frac{\hbar}{i}\nabla \rightarrow \frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}(\mathbf{r})$$

vector potential

Gaussian units

  - include energy density of magnetic field

$$\frac{B^2}{8\pi}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$

- Helmholtz free energy:

$$F[\psi, \mathbf{A}] = \int d^3r \left[\alpha\,|\psi(\mathbf{r})|^2 + \frac{\beta}{2}\,|\psi(\mathbf{r})|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}\right)\psi\right|^2 + \frac{B^2}{8\pi}\right]$$

- stationary solution with respect to ψ:

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}\right)^2 \psi + \alpha\,\psi + \beta\,|\psi|^2\psi = 0$$

first Ginzburg-Landau equation

- stationary solution of F with respect to $\mathbf{A}$:

$$\frac{\delta F}{\delta A_n(\mathbf{r})} = i\frac{q\hbar}{2m^*c}\psi^*(\mathbf{r})\frac{\partial\psi}{\partial r_n} - i\frac{q\hbar}{2m^*c}\frac{\partial\psi^*}{\partial r_n}\psi(\mathbf{r}) + \frac{q^2}{m^*c^2}\psi^*(\mathbf{r})\psi(\mathbf{r})A_n(\mathbf{r}) - \frac{1}{4\pi}\sum_{ij}\epsilon_{ijn}\frac{\partial}{\partial r_j}B_i(\mathbf{r}) = 0,$$

use Ampère's law

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{j}$$

to find

$$\mathbf{j} = \frac{c}{4\pi}\,\nabla \times \mathbf{B} = i\frac{q\hbar}{2m^*}\,([\nabla\psi^*]\psi - \psi^*\nabla\psi) - \frac{q^2}{m^*c}\,|\psi|^2\mathbf{A}$$

second Ginzburg-Landau equation

- special case of uniform ψ:

$$\mathbf{j} = -\frac{q^2|\psi|^2}{m^*c}\,\mathbf{A},$$

curl of Ampère's law then gives

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{4\pi}{c}\,\nabla \times \mathbf{j} = -\frac{4\pi\,q^2|\psi|^2}{m^*c^2}\,\mathbf{B},$$

with BAC-CAB rule get

$$\nabla^2\mathbf{B} - \nabla(\nabla\cdot\mathbf{B}) = \nabla^2\mathbf{B} = \frac{4\pi\, q^2|\psi|^2}{m^*c^2}\,\mathbf{B}$$

- contains another length scale: magnetic penetration depth

$$\lambda \equiv \sqrt{\frac{m^*c^2}{4\pi\, q^2|\psi|^2}},$$

get

$$\nabla^2\mathbf{B} = \frac{1}{\lambda^2}\mathbf{B}$$

- thus $\mathbf{B}$ *decays* into the superconductor on a length scale of λ: Meißner-Ochsenfeld effect
- two length scales: $\xi, \lambda \propto \frac{1}{|\psi|} \propto \frac{1}{\sqrt{-\alpha}} \propto \frac{1}{\sqrt{T_c - T}}$
- Their ratio $\kappa \equiv \lambda/\xi$, the Ginzburg-Landau parameter, is roughly temperature independent. For small κ, magnetic flux is easily expelled (type 1). For large κ, flux easily penetrates the superconductors, leading to vortex states (type 2).

see lecture notes

Ginzburg-Landau theory for superconductors

The Anderson-Higgs mechanism

- there are fluctuations of $\psi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$
- free-energy difference of superconducting and normal state:

$$F = \int d^3r \left[\alpha \, |\psi(\mathbf{r})|^2 + \frac{\beta}{2} \, |\psi(\mathbf{r})|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A} \right) \psi \right|^2 + \frac{B^2}{8\pi} - \frac{B_{\mathrm{appl}}^2}{8\pi} \right]$$

- For $T > T_c$ (Gaussian approximation):

$$F \cong \int d^3r \left[\alpha \, \psi^* \psi + \frac{1}{2m^*} \left| \frac{\hbar}{i} \nabla \psi \right|^2 + \frac{B^2}{8\pi} - \frac{B_{\mathrm{appl}}^2}{8\pi} \right] = \sum_{\mathbf{k}} \left(\alpha + \frac{\hbar^2 k^2}{2m^*} \right) \psi_{\mathbf{k}}^* \psi_{\mathbf{k}} + \sum_{\mathbf{k} \neq 0} \frac{B_{\mathbf{k}}^* \cdot B_{\mathbf{k}}}{8\pi}$$

- two massive modes $\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$ (like for neutral superfluid), magnetic-field modes decouple and are those of a free field (massless)

- For $T < T_c$ (Gaussian approximation):

$$F \cong \int d^3r \left[-2\alpha\, \delta\psi^2 + \frac{\hbar^2}{2m^*} \left(\frac{1}{i} \nabla \delta\psi \right)^* \cdot \frac{1}{i} \nabla \delta\psi - \frac{\hbar^2}{2m^*} \frac{\alpha}{\beta} \left(\nabla\phi - \frac{q}{\hbar c} \mathbf{A} \right)^* \cdot \left(\nabla\phi - \frac{q}{\hbar c} \mathbf{A} \right) + \frac{1}{8\pi} (\nabla \times \mathbf{A})^* \cdot (\nabla \times \mathbf{A}) \right]$$

- gauge invariance: physics invariant under gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\chi,$$
$$\psi' = e^{iq\chi/\hbar c}\, \psi,$$

$\chi(\mathbf{r}, t)$: arbitrary scalar field

thus for the phase

$$\phi' = \phi + \frac{q}{\hbar c} \chi$$

- choose gauge: $\phi' = 0$ so that

$$\mathbf{A}' = \mathbf{A} - \frac{\hbar c}{q} \nabla\phi$$

Gauge invariance is not a symmetry of the system but a redundancy of its description → χ does not describe physical fluctuations

then (drop the prime)

$$F[\delta\psi, \mathbf{A}] \cong \int d^3r \left[-2\alpha\, \delta\psi^2 + \frac{\hbar^2}{2m^*} \left(\frac{1}{i} \nabla \delta\psi \right)^* \cdot \frac{1}{i} \nabla \delta\psi - \frac{\alpha}{\beta} \frac{q^2}{2m^* c^2} \mathbf{A}^* \cdot \mathbf{A} + \frac{1}{8\pi} (\nabla \times \mathbf{A})^* \cdot (\nabla \times \mathbf{A}) \right]$$

$$= \sum_{\mathbf{k}} \left[\left(-2\alpha + \frac{\hbar^2 k^2}{2m^*} \right) \delta\psi_{\mathbf{k}}^* \delta\psi_{\mathbf{k}} - \frac{\alpha}{\beta} \frac{q^2}{2m^* c^2} \mathbf{A}_{\mathbf{k}}^* \cdot \mathbf{A}_{\mathbf{k}} + \frac{1}{8\pi} (\mathbf{k} \times \mathbf{A}_{\mathbf{k}})^* \cdot (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) \right]$$

- amplitude modes $\delta\psi$ are massive and decoupled: Higgs modes
- split electromagnetic field into longitudinal and transverse modes:

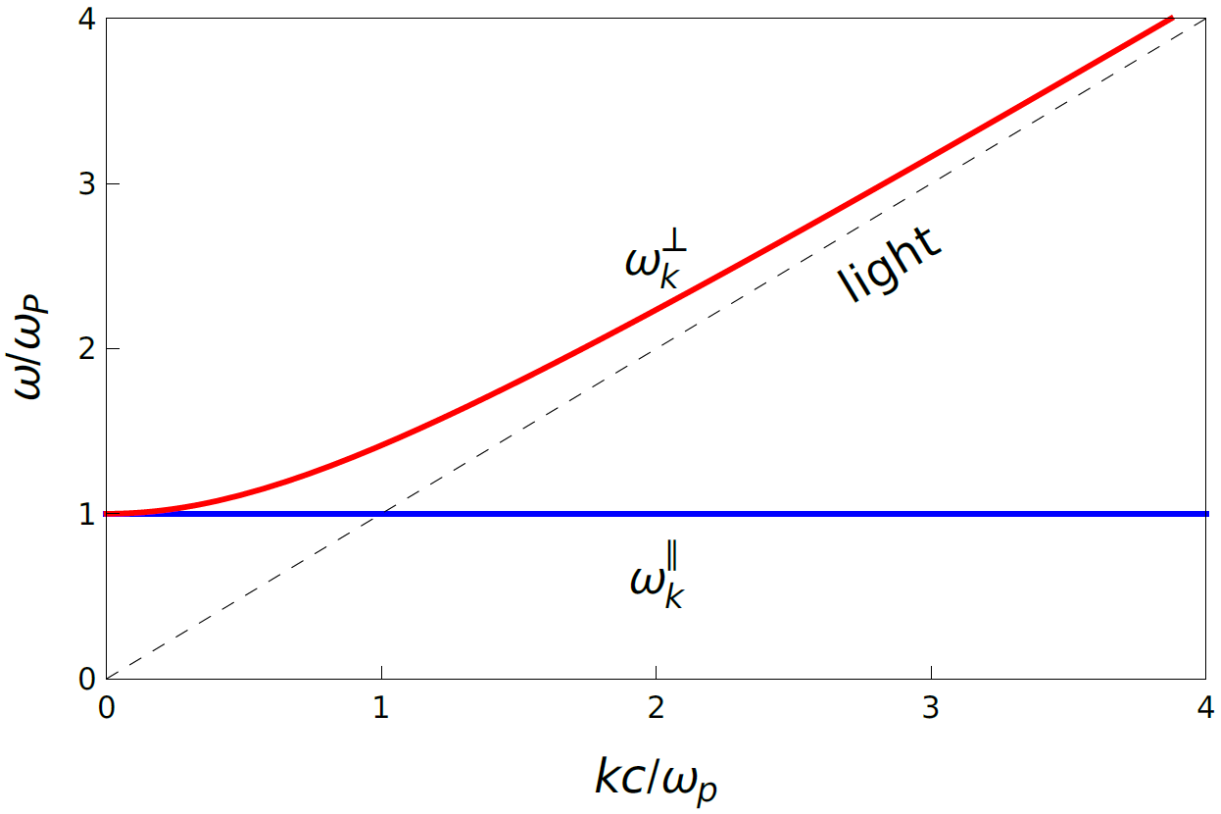
$$\mathbf{A}_{\mathbf{k}} = \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}) + \mathbf{A}_{\mathbf{k}} - \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}) \equiv \mathbf{A}_{\mathbf{k}}^{\parallel} + \mathbf{A}_{\mathbf{k}}^{\perp}, \qquad \hat{k} \equiv \frac{\mathbf{k}}{k}$$

then

$$F \cong \sum_{\mathbf{k}} \left[\left(-2\alpha + \frac{\hbar^2 k^2}{2m^*} \right) \delta\psi_{\mathbf{k}}^* \delta\psi_{\mathbf{k}} \underbrace{- \frac{\alpha}{\beta} \frac{q^2}{2m^* c^2} \mathbf{A}_{\mathbf{k}}^{\parallel *} \cdot \mathbf{A}_{\mathbf{k}}^{\parallel} - \frac{\alpha}{\beta} \frac{q^2}{2m^* c^2} \mathbf{A}_{\mathbf{k}}^{\perp *} \cdot \mathbf{A}_{\mathbf{k}}^{\perp}}_{\text{due to superconductivity}} + \underbrace{\frac{1}{8\pi} k^2 \mathbf{A}_{\mathbf{k}}^{\perp *} \cdot \mathbf{A}_{\mathbf{k}}^{\perp}}_{\text{free-field part}} \right]$$

- all three components of $\mathbf{A}$ appear and all are massive: Anderson-Higgs mechanism

with dispersion:



see lecture notes

with the plasma frequency $\omega_p \equiv \sqrt{-\frac{\alpha}{\beta}\frac{4\pi\, q^2}{m^*}}$

Ginzburg-Landau theory for superconductors

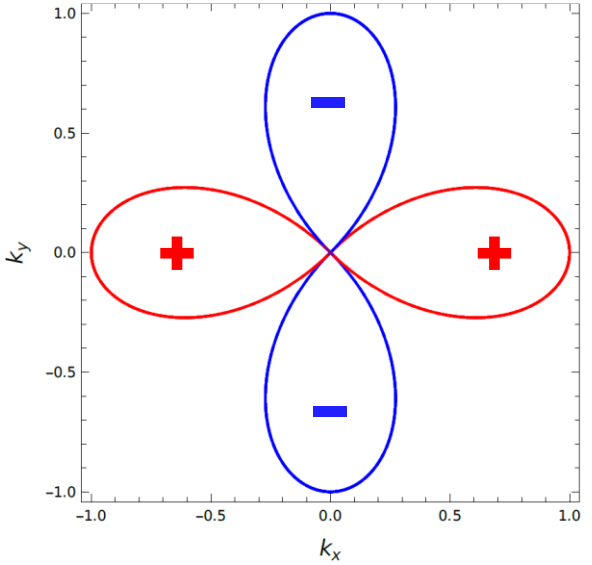
Ginzburg-Landau theory for nontrivial symmetries

- Ginzburg and Landau: expand F, include all symmetry-allowed terms!
  - crystal symmetries, in particular point group → representation theory of groups
  - time-reversal symmetry
  - gauge invariance
- so far, assumed a single complex order parameter field $\psi(\mathbf{r})$
- multiple order parameters $\psi_n(\mathbf{r})$, $n = 1, 2, \ldots$ are possible if
  - the order parameter has d complex components because it belongs to a d-dimensional irreducible representation (*irrep*) of the point group
  - multiple order parameters of different symmetry (different irreps) are accidentally close in energy

- **Procedure:** Step 1: determine how all order-parameter fields and other relevant fields transform under the point group and time reversal:
  - induction $\mathbf{B}$ is a *pseudovector* (rotates like a vector but is even under inversion)
  - $\mathbf{B}$ is odd under time reversal: $\mathbf{B} \mapsto -\mathbf{B}$
  - order parameters ψ_n can transform nontrivially under point group, e.g., $d_{x^2-y^2}$-wave pairing in the cuprates → encoded in irrep of irreducible tensor operators
  - $\psi_n \mapsto \psi_n^*$ under time reversal (→ inverts current $\mathbf{j}$)
  - gauge-covariant derivative

$$\mathbf{D} \equiv \nabla - i\frac{q}{\hbar c}\mathbf{A}$$

    is a *vector* under the point group and $\mathbf{D} \mapsto \mathbf{D}^*$ under time reversal



- Step 2: construct all combinations of $\psi_n, \mathbf{D}, \mathbf{B}, \ldots$ up to required order that have full symmetry (belong to *trivial* irrep), are even under time reversal, respect gauge invariance, and are real

Example: cubic superconductor with point group O_h, three-component order parameter $\boldsymbol{\psi} = (\psi_x, \psi_y, \psi_z) \in \mathbb{C}^3$ that transforms like a vector, i.e., belongs to irrep T_{1u}

- second-order term in F must contain one component of $\boldsymbol{\psi}^*$ and one of $\boldsymbol{\psi}$ due to gauge invariance; it is useful to consider the reduction of product of irreps

$$T_{1u} \otimes T_{1u} = A_{1g} \oplus E_g \oplus T_{1g} \oplus T_{2g}$$

  trivial irrep A_{1g} appears once → one possibility, which is

$$\boldsymbol{\psi}^* \cdot \boldsymbol{\psi} = \psi_x^* \psi_x + \psi_y^* \psi_y + \psi_z^* \psi_z$$

- fourth-order term:

$$T_{1u} \otimes T_{1u} \otimes T_{1u} \otimes T_{1u} = 4\, A_{1g} \oplus 3\, A_{2g} \oplus 7\, E_g \oplus 10\, T_{1g} \oplus 10\, T_{2g}$$

  expect 4 independent terms but one is zero, remaining:

$$(\boldsymbol{\psi}^* \cdot \boldsymbol{\psi})^2,$$
$$(\boldsymbol{\psi}^* \cdot \boldsymbol{\psi}^*)(\boldsymbol{\psi} \cdot \boldsymbol{\psi}),$$
$$\boldsymbol{\mu} \cdot \boldsymbol{\mu}$$

  $\boldsymbol{\mu}$ is a real time-reversal-odd pseudovector

with $\quad \boldsymbol{\mu} \equiv i\,(\boldsymbol{\psi}^* \times \boldsymbol{\psi}) = i \begin{pmatrix} \psi_y^* \psi_z - \psi_z^* \psi_y \\ \psi_z^* \psi_x - \psi_x^* \psi_z \\ \psi_x^* \psi_y - \psi_y^* \psi_x \end{pmatrix}$

- analogously: second order in order parameters, second order in derivatives: 3 terms

$$(\mathbf{D}\cdot\boldsymbol{\psi})^*(\mathbf{D}\cdot\boldsymbol{\psi}) = \left(D_x\psi_x + D_y\psi_y + D_z\psi_z\right)^*\left(D_x\psi_x + D_y\psi_y + D_z\psi_z\right),$$
$$(D_x\psi_x)^*(D_x\psi_x) + \left(D_y\psi_y\right)^*\left(D_y\psi_y\right) + (D_z\psi_z)^*(D_z\psi_z),$$
$$\left(D_y\psi_z\right)^*\left(D_y\psi_z\right) + \left(D_z\psi_y\right)^*\left(D_z\psi_y\right) + (D_z\psi_x)^*(D_z\psi_x) + (D_x\psi_z)^*(D_x\psi_z)$$
$$+\left(D_x\psi_y\right)^*\left(D_x\psi_y\right) + \left(D_y\psi_x\right)^*\left(D_y\psi_x\right)$$

- linear order in $\mathbf{B}$:

$$\mathbf{B}\cdot\boldsymbol{\mu} = i\left(B_x\psi_y^*\psi_z + B_y\psi_z^*\psi_x + B_z\psi_x^*\psi_y\right) + \text{c. c.}$$

nonzero if at least two components of $\boldsymbol{\psi}$ are nonzero and have a nontrivial phase difference ↔ broken time-reversal symmetry!

  - microscopic origin: spin-triplet pairing, Cooper-pair spin 1 couples to $\mathbf{B}$ by Zeeman coupling
  - $\mathbf{B}$ can *induce* time-reversal-symmetry-breaking order parameter

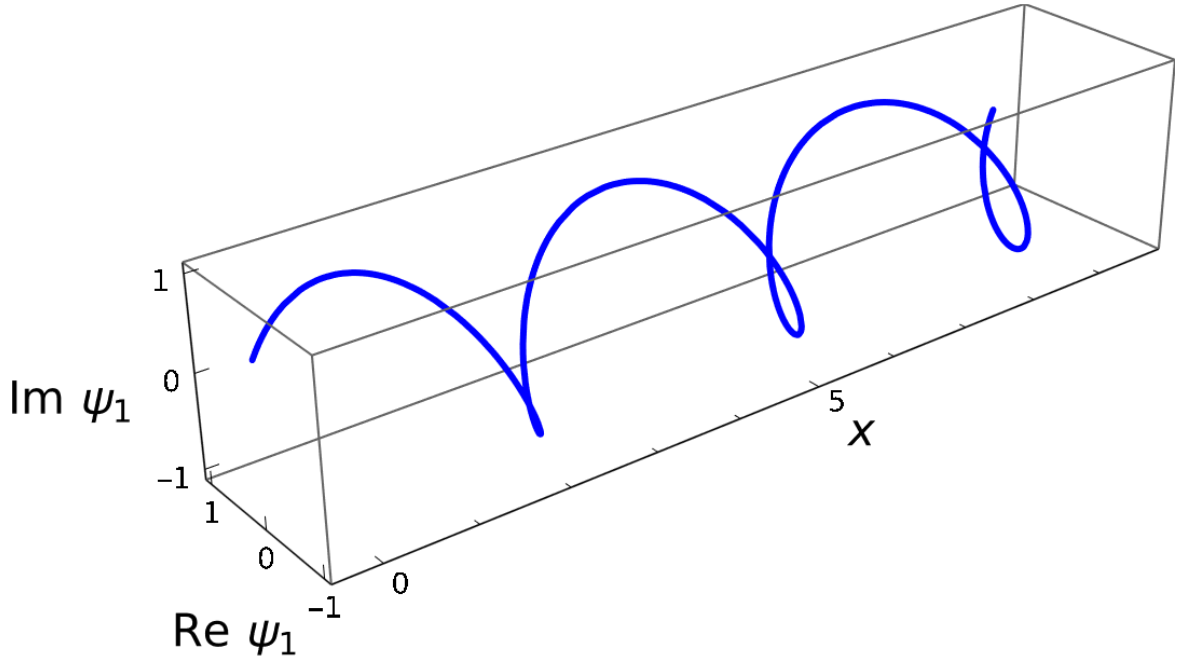
Example 2: broken inversion symmetry: point group O

In addition, a first-order term in derivatives is allowed:

$$\boldsymbol{\psi}^* \cdot (\mathbf{D} \times \boldsymbol{\psi}) = \psi_x^* D_y \psi_z + \psi_y^* D_z \psi_x + \psi_z^* D_x \psi_y + \text{c. c.}$$

It is a time-reversal-even *pseudoscalar*, i.e., *odd* under inversion (forbidden in O_h)

Such terms are called Lifshitz invariants. Ansatz $\boldsymbol{\psi}(\mathbf{r}) = \boldsymbol{\psi}_0 \, e^{i\mathbf{q}\cdot\mathbf{r}}$ shows that the equilibrium state has a finite modulation vector $\mathbf{q} \neq 0 \rightarrow$ helical superconducting state



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DRESDEN concept

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Ginzburg-Landau Theory of Superconductivity

- Introduction
- Landau theory of phase transitions
- Ginzburg-Landau theory for neutral superfluids
- Ginzburg-Landau theory for superconductors

For details of derivations and references see the lecture notes

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