Exercise Sheet 7 due 12 December

1. *Uncertainty relation*

Calculate the expectation value of the square of the position and the momentum operator (for $\langle n|p^2|n\rangle$ see last exercise) to verify the uncertainty relation for the eigenstates of a harmonic oscillator.

2. *Jordan algebra*

The product AB of two Hermitian operators A and B is Hermitian if and only if they commute, $[A,\, B]=0$. To make the product of two observables again an observable (i.e. Hermitian), we can introduce the symmetrized product $A*B := AB+BA$, which is obviously Hermitian. Show that the symmetrized product is commutative, $A*B = B*A$, but not associative $A*(B*C) \neq (A*B)*C$.

3. *Pauli matrices*

The Pauli matrices are defined as

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

i. Find the eigenvalues and (normalized) eigenvectors $|\chi_{z,n}\rangle$ of $\hat{\sigma}_z$.

ii. Find the eigenvalues and (normalized) eigenvectors $|\chi_{x,n}\rangle$ of $\hat{\sigma}_x$.

iii. Show by explicit calculation that $\sum_n |\chi_{x,n}\rangle\langle\chi_{x,n}|$ is the identity matrix.

iv. Determine the commutators between each pair of the Pauli matrices by explicit matrix multiplication. Write the result in terms of unit matrix and the Pauli matrices.

v. Calculate $\exp(\sigma_x)$ by transforming to the eigenbasis and, alternatively, by using the power series.