Exercise Sheet 8 due 19 December

1. *Spin*

Calculate the commutation relations of the operators $S_i := \hbar\hat{\sigma}_i/2$, where $\hat{\sigma}_i$ are the Pauli matrices of last week's exercise. Compare to the commutation relations for the operators of the orbital angular momenta $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

2. *Angular momentum operator in spherical coordinates*

Given

$$\begin{aligned}
\hat{L}_x &= i\hbar\left(+\sin\varphi\,\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\,\frac{\partial}{\partial\varphi}\right)\\
\hat{L}_y &= i\hbar\left(-\cos\varphi\,\frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\varphi\,\frac{\partial}{\partial\varphi}\right)\\
\hat{L}_z &= -i\hbar\,\frac{\partial}{\partial\varphi}
\end{aligned}$$

i. Show that $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \hbar\, e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\vartheta} + i\cot\vartheta\,\frac{\partial}{\partial\varphi}\right)$

ii. Show that $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)$

Hint: $\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z(\hat{L}_z - \hbar)$

3. *Solid harmonics*

Verify that the following polynomials of order l=2 in x, y and z

$$y^2-z^2,\, xy,\, xz,\, yz,\ \text{ and } 3x^2-r^2 = 2x^2-y^2-z^2 \tag{1}$$

solve the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) P(x,y,z) = 0\,.$$

Check that the polynomials are orthogonal, i.e., that

$$\int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-L}^{L} dz\, P_1(x,y,z)\, P_2(x,y,z) = 0$$

for $P_1 \neq P_2$ and $L\to\infty$. Note that the normalization integral (P_1=P_2) diverges for $L\to\infty$.

Also x^2-y^2 solves the Laplace equation. Write it as a linear combination of the polynomials (1).