Exercise Sheet 9 due 9 January

1. *spherical harmonics*

   i. Using the spherical harmonics for $l = 1$ derived in the lecture, calculate $|Y(\vartheta,\varphi)|^2$ for $Y(\vartheta,\varphi) = \frac{1}{\sqrt{3}}(Y_{1,-1}(\vartheta,\varphi) + Y_{1,0}(\vartheta,\varphi) + Y_{1,1}(\vartheta,\varphi))$.

   ii. Write the spherical harmonics for $l = 1$ in Cartesian coordinates, i.e., replace $\sin\vartheta\,\cos\varphi$ by x/r, $\sin\vartheta\,\sin\varphi$ by y/r, and $\cos\vartheta$ by z/r.

   iii. Write the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in the space of the spherical harmonics with $l{=}1$, i.e., the $3{\times}3$ matrices $\langle l{=}1, m|\hat{L}_i|l{=}1, m'\rangle$. Do the same for $\hat{L}_\pm$ and $\vec{L}^2$.

   iv. Using the ladder operators starting from

$$Y_{2,-2}(\vartheta,\varphi) = \sqrt{\frac{15}{32\pi}}\,(\sin\vartheta)^2\, e^{-2i\varphi}$$

   derive the spherical harmonics for $l{=}2$ with $m{>}{-2}$:

$$\begin{aligned}
Y_{2,-1}(\vartheta,\varphi) &= \sqrt{\frac{15}{8\pi}}\,\sin\vartheta\cos\vartheta\, e^{-i\varphi}\\
Y_{2,0}(\vartheta,\varphi) &= \sqrt{\frac{5}{16\pi}}\,(3(\cos\vartheta)^2 - 1)\\
Y_{2,1}(\vartheta,\varphi) &= -\sqrt{\frac{15}{8\pi}}\,\sin\vartheta\cos\vartheta\, e^{i\varphi}\\
Y_{2,2}(\vartheta,\varphi) &= \sqrt{\frac{15}{32\pi}}\,(\sin\vartheta)^2\, e^{2i\varphi}
\end{aligned}$$

   v. Write the spherical harmonics for $l{=}2$ in Cartesian coordinates and compare to the solid harmonics of order 2 from last exercise.