

Exercise Sheet 10

1. Gaunt coefficients

Verify the Gaunt coefficients $\langle Y_{l,m} | Y_{k,m-m'} Y_{l,m'} \rangle$ for p and d -shells in the basis of spherical harmonics:

$$l = 1 \quad k = 0 \quad \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$k = 2 \quad \frac{1}{\sqrt{20\pi}} \begin{pmatrix} -1 & +\sqrt{3} & -\sqrt{6} \\ -\sqrt{3} & 2 & -\sqrt{3} \\ -\sqrt{6} & +\sqrt{3} & -1 \end{pmatrix}$$

$$l = 2 \quad k = 0 \quad \frac{1}{\sqrt{4\pi}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$k = 2 \quad \frac{1}{\sqrt{196\pi}} \begin{pmatrix} -\sqrt{20} & +\sqrt{30} & -\sqrt{20} & 0 & 0 \\ -\sqrt{30} & \sqrt{5} & +\sqrt{5} & -\sqrt{30} & 0 \\ -\sqrt{20} & -\sqrt{5} & \sqrt{20} & -\sqrt{5} & -\sqrt{20} \\ 0 & -\sqrt{30} & +\sqrt{5} & \sqrt{5} & -\sqrt{30} \\ 0 & 0 & -\sqrt{20} & +\sqrt{30} & -\sqrt{20} \end{pmatrix}$$

$$k = 4 \quad \frac{1}{\sqrt{196\pi}} \begin{pmatrix} 1 & -\sqrt{5} & \sqrt{15} & -\sqrt{35} & \sqrt{70} \\ +\sqrt{5} & -4 & +\sqrt{30} & -\sqrt{40} & +\sqrt{35} \\ \sqrt{15} & -\sqrt{30} & 6 & -\sqrt{30} & \sqrt{15} \\ +\sqrt{35} & -\sqrt{40} & +\sqrt{30} & -4 & +\sqrt{5} \\ \sqrt{70} & -\sqrt{35} & \sqrt{15} & -\sqrt{5} & 1 \end{pmatrix}$$

Hint: For $\langle Y_{l,0} | Y_{k,0} Y_{l,0} \rangle$ perform the integral over the Legendre polynomials, for the other matrix elements, use the ladder operators.

Show that $\sqrt{4\pi} \text{Tr } g_{m,m}^{(k)} = \sqrt{4\pi} \sum_m \langle Y_{l,m} | Y_{k,0} Y_{l,m} \rangle = (2l+1) \delta_{k,0}$.