angular momentum algebra

$$\vec{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} \text{ where } J_i^\dagger = J_i \text{ and } [J_1, J_2] = i J_3 \text{ (cyclic)}$$

orthonormal eigensystem

$$\vec{J}^{\,2}|j,m\rangle = j(j+1)|j,m\rangle \qquad j \in \{0, 1/2, 1, 3/2, 2, \ldots\}$$

$$J_3|j,m\rangle = \qquad m|j,m\rangle \qquad m \in \{-j, -j+1, \ldots, j\}$$

ladder operators $J_\pm = J_1 \pm i J_2$

$$J_\pm|j,m\rangle = \sqrt{(j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

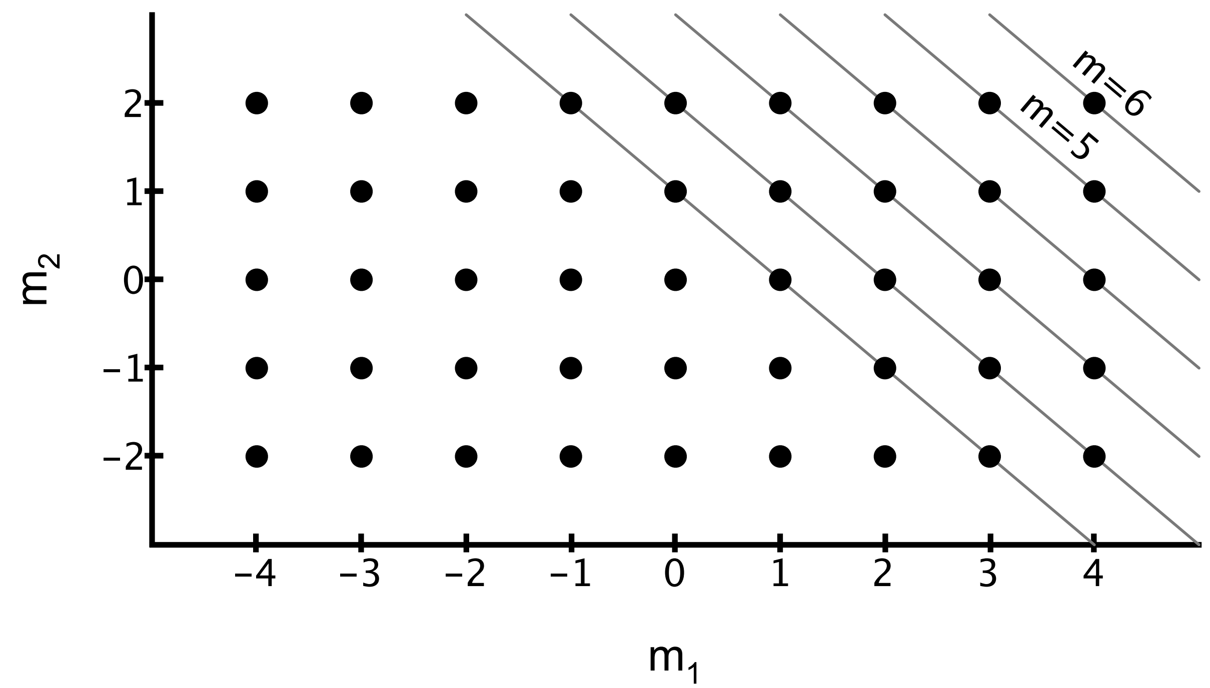
Condon-Shortley phase convention

adding angular momenta

$$\left[J_1^{(a)}, J_2^{(a)}\right] = i J_3^{(a)} \quad \text{and} \quad \left[J_i^{(a)}, J_k^{(b)}\right] = 0$$

tensor basis

$$\left(\vec{J}^{(1)}\right)^2 |j_1, m_1; j_2, m_2\rangle = j_1(j_1+1)|j_1, m_1; j_2, m_2\rangle$$
$$J_3^{(1)} |j_1, m_1; j_2, m_2\rangle = m_1|j_1, m_1; j_2, m_2\rangle$$
$$\left(\vec{J}^{(2)}\right)^2 |j_1, m_1; j_2, m_2\rangle = j_2(j_2+1)|j_1, m_1; j_2, m_2\rangle$$
$$J_3^{(2)} |j_1, m_1; j_2, m_2\rangle = m_2|j_1, m_1; j_2, m_2\rangle$$



$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

$$\vec{J}^2 |j, m; j_1; j_2\rangle = j(j+1)|j, m; j_1; j_2\rangle$$
$$J_3 |j, m; j_1; j_2\rangle = m|j, m; j_1; j_2\rangle$$

$$|j_1+j_2, j_1+j_2; j_1; j_2\rangle = |j_1, j_1; j_2, j_2\rangle$$

$$j = \{j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|\}$$

$$|j, m; j_1; j_2\rangle = \sum_{m_1+m_2=m} |j_1, m_1; j_2, m_2\rangle \underbrace{\langle j_1, m_1; j_2, m_2 | j, m; j_1; j_2\rangle}_{\text{Clebsch-Gordan coefficients}}$$

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

1/2×1/2

M	m_1	m_2	J=1	J=0
+1	+1/2	+1/2	1	
0	+1/2	−1/2	1/2	1/2
0	−1/2	+1/2	1/2	−1/2
−1	−1/2	−1/2	1	

1×1/2

M	m_1	m_2	J=3/2	J=1/2
+3/2	+1	+1/2	1	
+1/2	+1	−1/2	1/3	2/3
+1/2	0	+1/2	2/3	−1/3
−1/2	0	−1/2	2/3	1/3
−1/2	−1	+1/2	1/3	−2/3
−3/2	−1	−1/2	1	

1×1

M	m_1	m_2	J=2	J=1	J=0
+2	+1	+1	1		
+1	+1	0	1/2	1/2	
+1	0	+1	1/2	−1/2	
0	+1	−1	1/6	1/2	1/3
0	0	0	2/3	0	−1/3
0	−1	+1	1/6	−1/2	1/3
−1	0	−1	1/2	1/2	
−1	−1	0	1/2	−1/2	
−2	−1	−1	1		

2×1

M	m_1	m_2	J=3	J=2	J=1
+3	+2	+1	1		
+2	+2	0	1/3	2/3	
+2	+1	+1	2/3	−1/3	
+1	+2	−1	1/15	1/3	3/5
+1	+1	0	8/15	1/6	−3/10
+1	0	+1	2/5	−1/2	1/10
0	+1	−1	1/5	1/2	3/10
0	0	0	3/5	0	−2/5
0	−1	+1	1/5	−1/2	3/10
−1	0	−1	2/5	1/2	1/10
−1	−1	0	8/15	−1/6	−3/10
−1	−2	+1	1/15	−1/3	3/5
−2	−1	−1	2/3	1/3	
−2	−2	0	1/3	−2/3	
−3	−2	−1	1		

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

M	m_1	m_2	J=5/2	J=3/2	J=1/2
+5/2	+3/2	+1	1		
+3/2	+3/2	0	2/5	3/5	
+3/2	+1/2	+1	3/5	−2/5	
+1/2	+3/2	−1	1/10	2/5	1/2
+1/2	+1/2	0	3/5	1/15	−1/3
+1/2	−1/2	+1	3/10	−8/15	1/6
−1/2	+1/2	−1	3/10	8/15	1/6
−1/2	−1/2	0	3/5	−1/15	−1/3
−1/2	−3/2	+1	1/10	−2/5	1/2
−3/2	−1/2	−1	3/5	2/5	
−3/2	−3/2	0	2/5	−3/5	
−5/2	−3/2	−1	1		

2×1/2

M	m_1	m_2	J=5/2	J=3/2
+5/2	+2	+1/2	1	
+3/2	+2	−1/2	1/5	4/5
+3/2	+1	+1/2	4/5	−1/5
+1/2	+1	−1/2	2/5	3/5
+1/2	0	+1/2	3/5	−2/5
−1/2	0	−1/2	3/5	2/5
−1/2	−1	+1/2	2/5	−3/5
−3/2	−1	−1/2	4/5	1/5
−3/2	−2	+1/2	1/5	−4/5
−5/2	−2	−1/2	1	

3/2×1/2

M	m_1	m_2	J=2	J=1
+2	+3/2	+1/2	1	
+1	+3/2	−1/2	1/4	3/4
+1	+1/2	+1/2	3/4	−1/4
0	+1/2	−1/2	1/2	1/2
0	−1/2	+1/2	1/2	−1/2
−1	−1/2	−1/2	3/4	1/4
−1	−3/2	+1/2	1/4	−3/4
−2	−3/2	−1/2	1	

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

M	m_1	m_2	J=3	J=2	J=1	J=0
+3	+3/2	+3/2	1			
+2	+3/2	+1/2	1/2	1/2		
+2	+1/2	+3/2	1/2	−1/2		
+1	+3/2	−1/2	1/5	1/2	3/10	
+1	+1/2	+1/2	3/5	0	−2/5	
+1	−1/2	+3/2	1/5	−1/2	3/10	
0	+3/2	−3/2	1/20	1/4	9/20	1/4
0	+1/2	−1/2	9/20	1/4	−1/20	−1/4
0	−1/2	+1/2	9/20	−1/4	−1/20	1/4
0	−3/2	+3/2	1/20	−1/4	9/20	−1/4
−1	+1/2	−3/2	1/5	1/2	3/10	
−1	−1/2	−1/2	3/5	0	−2/5	
−1	−3/2	+1/2	1/5	−1/2	3/10	
−2	−1/2	−3/2	1/2	1/2		
−2	−3/2	−1/2	1/2	−1/2		
−3	−3/2	−3/2	1			

$$d^{1}_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

M	m_1	m_2	J=7/2	J=5/2	J=3/2	J=1/2
+7/2	+2	+3/2	1			
+5/2	+2	+1/2	3/7	4/7		
+5/2	+1	+3/2	4/7	−3/7		
+3/2	+2	−1/2	1/7	16/35	2/5	
+3/2	+1	+1/2	4/7	1/35	−2/5	
+3/2	0	+3/2	2/7	−18/35	1/5	
+1/2	+2	−3/2	1/35	6/35	2/5	2/5
+1/2	+1	−1/2	12/35	5/14	0	−3/10
+1/2	0	+1/2	18/35	−3/35	−1/5	1/5
+1/2	−1	+3/2	4/35	−27/70	2/5	−1/10
−1/2	+1	−3/2	4/35	27/70	2/5	1/10
−1/2	0	−1/2	18/35	3/35	−1/5	−1/5
−1/2	−1	+1/2	12/35	−5/14	0	3/10
−1/2	−2	+3/2	1/35	−6/35	2/5	−2/5
−3/2	0	−3/2	2/7	18/35	1/5	
−3/2	−1	−1/2	4/7	−1/35	−2/5	
−3/2	−2	+1/2	1/7	−16/35	2/5	
−5/2	−1	−3/2	4/7	3/7		
−5/2	−2	−1/2	3/7	−4/7		
−7/2	−2	−3/2	1			

2×2

M	m_1	m_2	J=4	J=3	J=2	J=1	J=0
+4	+2	+2	1				
+3	+2	+1	1/2	1/2			
+3	+1	+2	1/2	−1/2			
+2	+2	0	3/14	1/2	2/7		
+2	+1	+1	4/7	0	−3/7		
+2	0	+2	3/14	−1/2	2/7		
+1	+2	−1	1/14	3/10	3/7	1/5	
+1	+1	0	3/7	1/5	−1/14	−3/10	
+1	0	+1	3/7	−1/5	−1/14	3/10	
+1	−1	+2	1/14	−3/10	3/7	−1/5	
0	+2	−2	1/70	1/10	2/7	2/5	1/5
0	+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	0	18/35	0	−2/7	0	1/5
0	−1	+1	8/35	−2/5	1/14	1/10	−1/5
0	−2	+2	1/70	−1/10	2/7	−2/5	1/5
−1	+1	−2	1/14	3/10	3/7	1/5	
−1	0	−1	3/7	1/5	−1/14	−3/10	
−1	−1	0	3/7	−1/5	−1/14	3/10	
−1	−2	+1	1/14	−3/10	3/7	−1/5	
−2	0	−2	3/14	1/2	2/7		
−2	−1	−1	4/7	0	−3/7		
−2	−2	0	3/14	−1/2	2/7		
−3	−1	−2	1/2	1/2			
−3	−2	−1	1/2	−1/2			
−4	−2	−2	1				

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

continued fraction expansion

$$a = \lfloor a \rfloor + r = \lfloor a \rfloor + \frac{1}{1/r} = \lfloor a \rfloor + \frac{1}{\lfloor 1/r \rfloor + r_1} = \cdots$$

```
> bc -lq
define int(r) { auto s; s=scale; scale=0; r=r/1; scale=s; return r }
scale=40
a=4*a(1)
a; r=a-int(a)
3.1415926535897932384626433832795028841968
1/r; r=1/r-int(1/r)
7.0625133059310457697930051525705580427527
1/r; r=1/r-int(1/r)
15.9965944066857198889230604047552746009657
1/r; r=1/r-int(1/r)
1.0034172310133726034641471752879535964745
1/r; r=1/r-int(1/r)
292.6345910143954723785436957604105972678117
1/r; r=1/r-int(1/r)
1.5758180896283656987656107376683768305518
1/r; r=1/r-int(1/r)
1.7366595770643507716201135470042516251085
1/r; r=1/r-int(1/r)
1.3574791275843891875198494723817884952357
```

continued fraction expansion

```
#!/usr/bin/env bc -l
define int(r) {
  /* return integer part of r by temporariliy setting scale=0 */
  auto s; s=scale; scale=0
  r=r/1
  scale=s
  return r
}

scale=40; cforder=10
print "continued fraction expansion of pi: "
a=4*a(1) /* atan(x) */
print int(a), " "; r=a-int(a)
for (i=1; i<cforder; i++) { print int(1/r), " "; r=1/r-int(1/r) }
print "\n"

print "continued fraction expansion of e:  "
a=e(1) /* exp(x) */
print int(a), " "; r=a-int(a)
for (i=1; i<cforder; i++) { print int(1/r), " "; r=1/r-int(1/r) }
print "\n"

/* continued fraction expansion of pi: 3 7 15 1 292 1 1 1 2 1
   continued fraction expansion of e:  2 1 2 1 1 4 1 1 6 1      */
```